PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1963 **255**, 389-416
doi: 10.1098/rsta.1963.0008

PROPAGATION OF ELECTROMAGNETIC WAVES IN CIRCULAR RODS IN TORSION

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(*Communicated by A. E. Green, F.R.S.—Received* 23 *February* 1962—
Revised 13 *July* 1962)

CONTENTS

Invariance considerations are employed to write down constitutive equations governing the propagation of electromagnetic waves in isotropic materials with a centre of symmetry which are subject to a static deformation. It is assumed that the dielectric displacement and magnetic induction vectors are linear functions of the electric and magnetic field intensities, respectively, but are general polynomial functions in the quantities which specify the deformation.

The theory is employed to examine propagation along circular cylindrical rods in torsion. Rotating waves are produced whose speed of propagation and rate of rotation depend upon the magnitude of the deformation and the properties of the material. The nature of these waves is examined for the general case where there is no restriction either upon the amount of torsion or upon the magnitude of the effect. When the amount of torsion, or the dependence of the effect upon deformation is small, solutions can be obtained based upon those for the propagation of waves in undeformed materials.

1. Introduction

The formulation of constitutive equations in continuum physics has been considered in general terms by Pipkin & Rivlin (1959) and by Rivlin (1960). These relations characterize physical properties of materials and are restricted by the assumption that they must be unaltered by a simultaneous rotation of the reference frame and the physical system which they describe. Further restrictions are imposed by the symmetry of the material being examined. Such constitutive equations have been applied to problems involving electrical conduction in deformed elastic materials by Pipkin & Rivlin (1960, 1961).

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In the present paper we consider the propagation of electromagnetic waves in deformed elastic materials. It is assumed that the electric displacement vector **D** is a linear function of the electric field intensity **E** and that the magnetic induction vector **B** is a linear function of the magnetic field intensity **H**. In addition, the vectors **D** and **B** involve the quantities defining the deformation non-linearly. Attention is confined to isotropic materials possessing a centre of symmetry. It is further assumed that the deformation is unaffected by the presence of electric and magnetic fields and that free charges and currents are zero. In these circumstances the relations connecting **D** and **B** with **E** and **H** resemble those for an undeformed aeolotropic material, the nature of the aeolotropy at each point depending upon the deformation. When the expressions for **D** and **B** are introduced into Maxwell's equations there result six linear differential equations for the determination of the components of **E** and **H**.

The theory is employed in the present instance to examine the propagation of electromagnetic waves in twisted circular cylindrical rods. The analysis in cylindrical polar coordinates leads to a pair of second-order differential equations involving two of the field variables. Each of these equations reduces to the standard Bessel equation when the material is undeformed. The speed of propagation of the wave is determined from a secular equation derived from the boundary conditions, and this aspect is illustrated by the examples of the circular and coaxial waveguides. For a given mode of transmission the secular equation yields a pair of waves propagated with different speeds; these combine to give a wave which rotates during propagation.

Devices using such a rotating wave are of importance in micro-wave transmission (see, for example, Katz 1959). In these applications the rotation is usually produced by introducing ferrites into the waveguide and applying a static magnetic field. Mathematical aspects of waveguides of this kind have been investigated by Kales, Chait & Sakiotis (1953), Gamo (1953) and others. The rotating waves in these ferrite-filled waveguides are non-reciprocal, that is, if the direction of propagation is reversed, the direction of rotation of the wave is unaltered. For the materials examined in the present paper the wave is reciprocal and if the direction of propagation is reversed the direction of rotation is also reversed. The wave pattern then retraces its original path.

In the final sections of the paper, the theory is specialized to the cases where the angle of torsion is small (§§9 to 11), and where the angle of torsion is large but the dependence of electromagnetic properties upon deformation is small (§§12 to 14). In both cases, modes of transmission may be distinguished which differ only slightly from transverse magnetic or transverse electric waves in the undeformed material and to the order of approximation considered, the wave is transmitted without attenuation or distortion. The rate of rotation of the wave differs for these two types of propagation, but in the case where the amount of torsion is small it depends in a simple manner upon the constants describing the variation of electric and magnetic properties with deformation. The perturbation procedures of §§9 to 14 may, in principle, be extended to obtain higher-order approximations to the field components by assuming power series expansions either in the small parameter defining the torsion (§§9 to 11) or that describing the dependence upon deformation (§§12 to 14), and evaluating successive terms of the series by methods analogous to these used in finite elasticity (see, for example, Green & Adkins 1960; Rivlin 1953; Rivlin & Topakoglu 1954).

For convenience, the theory has been related to elastic materials. Since, however, the electromagnetic constitutive equations are unaffected by the stress deformation relations of the material, the analysis applies unchanged for any deformable material held in a state of static deformation.

2. Electromagnetic waves in deformed elastic materials

We consider a material which is isotropic in its undeformed state and possesses a centre of symmetry. The material is held in a given state of deformation and an electric field **E** and magnetic field **H** are applied to it. We assume that the resulting electric displacement field **D** and magnetic induction field **B**, at the point x^i of a curvilinear co-ordinate system x, are related to **E** and **H** by

$$D^i = \alpha^i_j E^j \quad \text{and} \quad B^i = \beta^i_j H^j, \tag{2·1}$$

where

$$\left.\begin{aligned} \alpha^i_j &= \alpha_0 \delta^i_j + \alpha_1 c^i_j + \alpha_2 c^i_k c^k_j, \\ \beta^i_j &= \beta_0 \delta^i_j + \beta_1 c^i_j + \beta_2 c^i_k c^k_j, \end{aligned}\right\} \tag{2·2}$$

E^i, H^i, D^i and B^i are the contravariant components of **E**, **H**, **D** and **B** respectively in the system x and c^i_j is defined by

$$c^i_j = G^{rs} \frac{\partial x^i}{\partial X^r} \frac{\partial x^k}{\partial X^s} g_{kj} - G^{ik} g_{kj}. \tag{2·3}$$

In (2·3), X^i are the co-ordinates in the undeformed material of the particle which is at x^i in the deformed state, G^{rs} is the contravariant metric tensor associated with the curvilinear system x^i at the point X^i and g_{kj} is the covariant metric tensor for the system at the point x^i. In (2·2) δ^i_j denotes the Kronecker delta and α_0, α_1, α_2, β_0, β_1 and β_2 are polynomial functions of the three independent invariants

$$c^i_i, \quad \tfrac{1}{2}(c^i_i c^j_j - c^i_j c^j_i), \quad |c^i_j| \tag{2·4}$$

of c^i_j.

The relations (2·1) and (2·2) imply that in the deformed state the material becomes curvilinearly aeolotropic as regards its electromagnetic properties, the nature of this aeolotropy depending upon the deformation. In general, when α^i_j and β^i_j are different functions, the extent of the aeolotropy is different for the electric and magnetic properties. When the deformation is uniform, α^i_j and β^i_j are constants and the material behaves as a uniformly (rectilinearly) aeolotropic medium.

In the absence of free electric currents and charges, the electromagnetic field equations take the form

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}. \tag{2·5}$$

For a sinusoidal wave, of angular frequency ω, we take

$$\left.\begin{aligned} \mathbf{E} &= \mathscr{R}\bar{\mathbf{e}} \exp(\mathrm{i}\omega t), & \mathbf{H} &= \mathscr{R}\bar{\mathbf{h}} \exp(\mathrm{i}\omega t), \\ \mathbf{D} &= \mathscr{R}\bar{\mathbf{d}} \exp(\mathrm{i}\omega t) \quad \text{and} & \mathbf{B} &= \mathscr{R}\bar{\mathbf{b}} \exp(\mathrm{i}\omega t), \end{aligned}\right\} \tag{2·6}$$

where $\bar{\mathbf{e}}$, $\bar{\mathbf{h}}$, $\bar{\mathbf{d}}$ and $\bar{\mathbf{b}}$ are complex vectors dependent on x^i and independent of time. Adopting the usual convention, we shall omit the symbol $\mathscr{R}$ in the following discussion. Introducing (2·6) into (2·5), we obtain

$$\operatorname{curl} \bar{\mathbf{e}} + \mathrm{i}\omega \bar{\mathbf{b}} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \bar{\mathbf{h}} - \mathrm{i}\omega \bar{\mathbf{d}} = \mathbf{0}. \tag{2·7}$$

In the curvilinear co-ordinate system x, equations (2·7) take the form

$$\left.\begin{aligned} g^{-\frac{1}{2}} g_{kn} \epsilon^{ijn} \bar{e}^k{}_{,j} + \mathrm{i}\omega \bar{b}^i &= 0 \\ \text{and} \qquad g^{-\frac{1}{2}} g_{kn} \epsilon^{ijn} \bar{h}^k{}_{,j} - \mathrm{i}\omega \bar{d}^i &= 0, \end{aligned}\right\} \tag{2·8}$$

where $g = |g_{ij}|$, ϵ^{ijn} is the alternating symbol (defined by $\epsilon^{ijn} = 1$ or -1 accordingly as ijn is an even or odd permutation of 123 and 0 otherwise), and $_{,j}$ denotes covariant differentiation with respect to x^j. From (2·1), (2·6) and (2·8)

$$\left.\begin{aligned} g^{-\frac{1}{2}} g_{kn} \epsilon^{ijn} \bar{e}^k{}_{,j} + \mathrm{i}\omega \beta^i_j \bar{h}^j &= 0 \\ \text{and} \qquad g^{-\frac{1}{2}} g_{kn} \epsilon^{ijn} \bar{h}^k{}_{,j} - \mathrm{i}\omega \alpha^i_j \bar{e}^j &= 0. \end{aligned}\right\} \tag{2·9}$$

3. Torsion

We now take as the curvilinear co-ordinate system x a cylindrical polar co-ordinate system (r, θ, z) and suppose that a particle of the material, which in its undeformed state is at (R, Θ, Z) in this system, moves to (r, θ, z) in the deformed state. Then

$$(X^1, X^2, X^3) = (R, \Theta, Z), \quad (x^1, x^2, x^3) = (r, \theta, z), \tag{3·1}$$

and the metric tensors g_{ij}, G^{ij} are given by

$$\left.\begin{aligned} g_{11} = g_{33} = 1, \quad g_{22} = r^2, \quad g_{ij} = 0 \quad (i \neq j), \\ g^{\frac{1}{2}} = r, \end{aligned}\right\} \tag{3·2}$$

and
$$G^{11} = G^{33} = 1, \quad G^{22} = 1/R^2, \quad G^{ij} = 0 \quad (i \neq j). \tag{3·3}$$

Also, we have

$$\|\bar{e}^i{}_{,j}\| = \left\| \begin{matrix} \dfrac{\partial \bar{e}^1}{\partial r} & -r\bar{e}^2 + \dfrac{\partial \bar{e}^1}{\partial \theta} & \dfrac{\partial \bar{e}^1}{\partial z} \\ \dfrac{\bar{e}^2}{r} + \dfrac{\partial \bar{e}^2}{\partial r} & \dfrac{\bar{e}^1}{r} + \dfrac{\partial \bar{e}^2}{\partial \theta} & \dfrac{\partial \bar{e}^2}{\partial z} \\ \dfrac{\partial \bar{e}^3}{\partial r} & \dfrac{\partial \bar{e}^3}{\partial \theta} & \dfrac{\partial \bar{e}^3}{\partial z} \end{matrix} \right\|. \tag{3·4}$$

We may obtain $\bar{h}^i{}_{,j}$ by replacing $\bar{e}$ by $\bar{h}$ in (3·4).

We consider a rod (or tube) of circular cross-section which is subjected to a simple torsion. We take the z axis of the cylindrical polar co-ordinate system (r, θ, z) along the axis of the rod, and define the deformation by

$$r = r(R), \quad \theta = \Theta + \tau Z, \quad z = Z, \tag{3·5}$$

where τ is a constant.

Using a prime to denote differentiation with respect to R, we obtain from (2·3), with (3·1) to (3·3) and (3·5)

$$\|c^i_j\| = \left\| \begin{matrix} r'^2 - 1 & 0 & 0 \\ 0 & \tau^2 r^2 & \tau \\ 0 & \tau r^2 & 0 \end{matrix} \right\|. \tag{3·6}$$

From (3·6), we obtain

$$\|c^i_k c^k_j\| = \left\| \begin{matrix} (r'^2 - 1)^2 & 0 & 0 \\ 0 & \tau^2 r^2 (1 + \tau^2 r^2) & \tau^3 r^2 \\ 0 & \tau^3 r^4 & \tau^2 r^2 \end{matrix} \right\| \tag{3·7}$$

and

$$\left.\begin{aligned} c^i_i &= r'^2+\tau^2r^2-1,\\ \tfrac{1}{2}(c^i_ic^j_j-c^i_jc^j_i) &= \tau^2r^2(r'^2-2),\\ |c^i_j| &= -(r'^2-1)\,\tau^2r^2.\end{aligned}\right\} \tag{3·8}$$

Introducing (3·6) and (3·7) into (2·2), we obtain

$$\|\alpha^i_j\| = \left\|\begin{matrix} \alpha_0+\alpha_1(r'^2-1)+\alpha_2(r'^2-1)^2 & 0 & 0\\ 0 & \alpha_0+\tau^2r^2[\alpha_1+\alpha_2(1+\tau^2r^2)] & \tau(\alpha_1+\alpha_2\tau^2r^2)\\ 0 & \tau r^2(\alpha_1+\alpha_2\tau^2r^2) & \alpha_0+\alpha_2\tau^2r^2\end{matrix}\right\|. \tag{3·9}$$

4. Wave propagation in a twisted rod

We now consider a plane electromagnetic wave which is propagated along the deformed rod. Since the wave is propagated in the z-direction and the vector $\bar{\mathbf{e}}$ is a single valued function of position we may assume that it can be expressed in the form

$$\bar{\mathbf{e}} = \sum_{n=-\infty}^{\infty} \mathbf{e}^{(n)}\exp[\mathrm{i}(n\theta-p_nz)], \tag{4·1}$$

where $\mathbf{e}^{(n)}$ are functions of r only and p_n are constants which may be real or complex. We therefore consider solutions of (2·9) of the form

$$\bar{\mathbf{e}} = \mathbf{e}\exp[\mathrm{i}(n\theta-pz)], \quad (p_n=p), \tag{4·2}$$

where $\mathbf{e}$ is a function of r only and n is a positive or negative integer. The vectors $\bar{\mathbf{h}}$, $\bar{\mathbf{d}}$ and $\bar{\mathbf{b}}$ may be expressed in similar form.

Employing (3·2), (3·4) and (3·9) in (2·9), we obtain

$$\left.\begin{aligned} \frac{\partial\bar{e}^3}{\partial\theta}-r^2\frac{\partial\bar{e}^2}{\partial z}+\mathrm{i}\omega r\,\beta^1_1\,\bar{h}^1 &= 0,\\ \frac{\partial\bar{e}^1}{\partial z}-\frac{\partial\bar{e}^3}{\partial r}+\mathrm{i}\omega r(\beta^2_2\bar{h}^2+\beta^2_3\bar{h}^3) &= 0,\\ \frac{\partial}{\partial r}(r^2\bar{e}^2)-\frac{\partial\bar{e}^1}{\partial\theta}+\mathrm{i}\omega r(\beta^3_2\bar{h}^2+\beta^3_3\bar{h}^3) &= 0,\end{aligned}\right\} \tag{4·3}$$

and

$$\left.\begin{aligned} \frac{\partial\bar{h}^3}{\partial\theta}-r^2\frac{\partial\bar{h}^2}{\partial z}-\mathrm{i}\omega r\,\alpha^1_1\,\bar{e}^1 &= 0,\\ \frac{\partial\bar{h}^1}{\partial z}-\frac{\partial\bar{h}^3}{\partial r}-\mathrm{i}\omega r(\alpha^2_2\bar{e}^2+\alpha^2_3\bar{e}^3) &= 0,\\ \frac{\partial}{\partial r}(r^2\bar{h}^2)-\frac{\partial\bar{h}^1}{\partial\theta}-\mathrm{i}\omega r(\alpha^3_2\bar{e}^2+\alpha^3_3\bar{e}^3) &= 0.\end{aligned}\right\} \tag{4·4}$$

The physical components of $\mathbf{e}$ and $\mathbf{h}$ are given by

$$\left.\begin{aligned} (e_r,e_\theta,e_z) = (e^1,re^2,e^3),\ (h_r,h_\theta,h_z) = (h^1,rh^2,h^3),\\ \text{with}\quad \bar{e}^k = e^k\exp[\mathrm{i}(n\theta-pz)],\ \bar{h}^k = h^k\exp[\mathrm{i}(n\theta-pz)].\end{aligned}\right\} \tag{4·5}$$

Introducing (4·5) into equations (4·3) and (4·4) we obtain

$$\left.\begin{aligned} pre_\theta+\omega r\beta^1_1h_r &= -ne_z,\\ pe_r-\omega\beta^2_2h_\theta &= \mathrm{i}(\mathrm{d}e_z/\mathrm{d}r)+\omega r\beta^2_3h_z,\\ \mathrm{d}(re_\theta)/\mathrm{d}r-\mathrm{i}ne_r+\mathrm{i}\omega(\beta^3_2h_\theta+r\beta^3_3h_z) &= 0,\end{aligned}\right\} \tag{4·6}$$

and

$$\left.\begin{aligned} \omega r\alpha_1^1 e_r - prh_\theta &= nh_z, \\ \omega\alpha_2^2 e_\theta + ph_r &= -\omega r\alpha_3^2 e_z + \mathrm{i}(\mathrm{d}h_z/\mathrm{d}r), \\ \mathrm{d}(rh_\theta)/\mathrm{d}r - \mathrm{i}nh_r - \mathrm{i}\omega(\alpha_2^3 e_\theta + r\alpha_3^3 e_z) &= 0. \end{aligned}\right\} \tag{4·7}$$

From the second of (4·6) and the first of (4·7) it follows that

$$\left.\begin{aligned} re_r &= \{\mathrm{i}pr(\mathrm{d}e_z/\mathrm{d}r) - \omega(n\beta_2^2 - pr^2\beta_3^2)\, h_z\}/\Delta_1, \\ rh_\theta &= \{\mathrm{i}\omega r\alpha_1^1(\mathrm{d}e_z/\mathrm{d}r) - (np - \omega^2 r^2\alpha_1^1\beta_3^2)\, h_z\}/\Delta_1, \end{aligned}\right\} \tag{4·8}$$

where

$$\Delta_1 = p^2 - \omega^2\alpha_1^1\beta_2^2. \tag{4·9}$$

Similarly from the first of (4·6) and the second of (4·7), we obtain

$$\left.\begin{aligned} re_\theta &= -\{(np - \omega^2 r^2\beta_1^1\alpha_3^2)\, e_z + \mathrm{i}\omega r\beta_1^1(\mathrm{d}h_z/\mathrm{d}r)\}/\Delta_2, \\ rh_r &= \{\omega(n\alpha_2^2 - pr^2\alpha_3^2)\, e_z + \mathrm{i}pr(\mathrm{d}h_z/\mathrm{d}r)\}/\Delta_2, \end{aligned}\right\} \tag{4·10}$$

where

$$\Delta_2 = p^2 - \omega^2\alpha_2^2\beta_1^1. \tag{4·11}$$

Introducing the expressions (4·8) and (4·10) into the third of equations (4·6) and (4·7) we obtain

$$\mathrm{i}\left(A_1\frac{\mathrm{d}^2 e_z}{\mathrm{d}r^2} + A_2\frac{\mathrm{d}e_z}{\mathrm{d}r} + A_3 e_z\right) + B_2\frac{\mathrm{d}h_z}{\mathrm{d}r} + B_3 h_z = 0, \tag{4·12}$$

$$C_2\frac{\mathrm{d}e_z}{\mathrm{d}r} + C_3 e_z + \mathrm{i}\left(D_1\frac{\mathrm{d}^2 h_z}{\mathrm{d}r^2} + D_2\frac{\mathrm{d}h_z}{\mathrm{d}r} + D_3 h_z\right) = 0, \tag{4·13}$$

where, since $\alpha_2^3 = r^2\alpha_3^2$, $\beta_2^3 = r^2\beta_3^2$,

$$\left.\begin{aligned} A_1 &= \omega r^2\alpha_1^1\Delta_1\Delta_2, \\ A_2 &= \omega\Delta_1^2\Delta_2 r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r\alpha_1^1}{\Delta_1}\right), \\ A_3 &= \omega\Delta_1^2\{\omega^2 r^2\beta_1^1(\alpha_2^2\alpha_3^3 - \alpha_3^2\alpha_2^3) - (n^2\alpha_2^2 - 2np\alpha_2^3 + p^2r^2\alpha_3^3)\}, \\ B_2 &= \omega^2\Delta_1 r\{p^2(\alpha_1^1\beta_2^3 - \alpha_2^3\beta_1^1) - np(\alpha_1^1\beta_2^2 - \alpha_2^2\beta_1^1) + \omega^2\alpha_1^1\beta_1^1(\alpha_2^3\beta_2^2 - \alpha_2^2\beta_2^3)\}, \\ B_3 &= -\Delta_1^2\Delta_2 r\,\mathrm{d}[(np - \omega^2\alpha_1^1\beta_2^3)/\Delta_1]/\mathrm{d}r, \end{aligned}\right\} \tag{4·14}$$

and

$$\left.\begin{aligned} C_2 &= -\Delta_2 B_2/\Delta_1, \\ C_3 &= -\Delta_1\Delta_2^2 r\,\mathrm{d}[(np - \omega^2\beta_1^1\alpha_2^3)/\Delta_2]/\mathrm{d}r, \\ D_1 &= -\omega r^2\beta_1^1\Delta_1\Delta_2, \\ D_2 &= -\omega\Delta_1\Delta_2^2 r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r\beta_1^1}{\Delta_2}\right), \\ D_3 &= -\omega\Delta_2^2\{\omega^2 r^2\alpha_1^1(\beta_2^2\beta_3^3 - \beta_3^2\beta_2^3) - (n^2\beta_2^2 - 2np\beta_2^3 + p^2r^2\beta_3^3)\}. \end{aligned}\right\} \tag{4·15}$$

From the system (4·12), (4·13) we may derive for h_z four independent solutions, which we may denote by R_μ and the general solution may be written as

$$h_z = \sum_{\mu=1}^{4} K_\mu R_\mu, \tag{4·16}$$

where K_μ are arbitrary constants. The functions R_μ depend also upon ω, n, p and τ so that we may write

$$R_\mu = R_\mu(r) = R_\mu(r;\, \omega, \tau, n;\, p). \tag{4·17}$$

When h_z has been determined, e_z may be found by eliminating e''_z, e'_z from (4·12), (4·13) and the equation

$$C_2 e''_z + (C'_2 + C_3)\, e'_z + C'_3 e_z + \mathrm{i}\,\mathrm{d}(D_1 h''_z + D_2 h'_z + D_3 h_z)/\mathrm{d}r = 0, \tag{4·18}$$

which is obtained by differentiating (4·13). The prime denotes differentiation with respect to r. This procedure yields

$$\mathrm{i}\begin{vmatrix} A_1 & A_2 & A_3 \\ 0 & C_2 & C_3 \\ C_2 & C'_2+C_3 & C'_3 \end{vmatrix} e_z = -\begin{vmatrix} \mathrm{i}A_1 & \mathrm{i}A_2 & B_2 h'_z + B_3 h_z \\ 0 & C_2 & \mathrm{i}(D_1 h''_z + D_2 h'_z + D_3 h_z) \\ C_2 & C'_2 + C_3 & \mathrm{i}\,\mathrm{d}(D_1 h''_z + D_2 h'_z + D_3 h_z)/\mathrm{d}r \end{vmatrix}. \tag{4·19}$$

Alternatively we may derive from (4·12), (4·13) four independent solutions for e_z. Denoting these by

$$S_\mu = S_\mu(r) = S_\mu(r;\, \tau, \omega, n;\, p), \tag{4·20}$$

we have

$$e_z = \sum_{\mu=1}^{4} L_\mu S_\mu, \tag{4·21}$$

where L_μ are further arbitrary constants. When e_z has been determined, h_z may be evaluated from the equation

$$\mathrm{i}\begin{vmatrix} D_1 & D_2 & D_3 \\ 0 & B_2 & B_3 \\ B_2 & B'_2+B_3 & B'_3 \end{vmatrix} h_z = -\begin{vmatrix} \mathrm{i}D_1 & \mathrm{i}D_2 & C_2 e'_z + C_3 e_z \\ 0 & B_2 & \mathrm{i}(A_1 e''_z + A_2 e'_z + A_3 e_z) \\ B_2 & B'_2 + B_3 & \mathrm{i}\,\mathrm{d}(A_1 e''_z + A_2 e'_z + A_3 e_z)/\mathrm{d}r \end{vmatrix}, \tag{4·22}$$

analogous to (4·19).

By introducing the expression (4·19) for e_z into (4·13) we may obtain a fourth-order differential equation for h_z. Similarly from (4·22) and (4·12) we may derive a fourth-order differential equation for e_z. These equations for e_z and h_z are, of course, identical.

Since R_μ and S_μ are related by (4·19) or (4·22), we may, without loss of generality, choose S_μ in such a way that, in the general solution of equations (4·12) and (4·13) given by (4·16) and (4·21), we have $L_\mu = K_\mu$. Then, equations (4·16) and (4·21) may be rewritten as

$$\left.\begin{aligned} h_z &= \sum_{\mu=1}^{4} K_\mu R_\mu \\ \text{and}\qquad e_z &= \sum_{\mu=1}^{4} K_\mu S_\mu. \end{aligned}\right\} \tag{4·23}$$

5. Behaviour of solutions on the axis

We confine attention to situations in which r' may be expressed as a polynomial in r^2. These include as a special case the torsion of an incompressible solid rod for which $r' = 1$; more generally, for a solid rod we assume $r' = 1 + O(r^2)$. The invariants (3·8) and the coefficients α^i_j, β^i_j may then be expressed as polynomials in r^2 and the coefficients A_μ, B_μ, C_μ and D_μ in (4·12) and (4·13) become polynomials in r.

In these circumstances, we may assume that series solutions of (4·12) and (4·13) of the forms

$$R_\mu = \sum_{q=0}^{\infty} c_q^{(\mu)} r^{k+q}, \quad S_\mu = \sum_{q=0}^{\infty} d_q^{(\mu)} r^{l+q} \quad (\mu = 1, 2, 3, 4) \tag{5·1}$$

exist with a finite range of convergence. The behaviour of the functions R_μ, S_μ at $r = 0$ is determined by the values of k and l in (5·1) and equations for these quantities are derived by considering the coefficients of the lowest powers of r occurring when the series (5·1) are introduced into (4·12) and (4·13).

Since α^i_j and β^i_j are polynomials in r^2, we may write

$$\|\alpha^i_j\| = \begin{Vmatrix} a_0 + a_1 r^2 & 0 & 0 \\ 0 & a_0 + a_{22} r^2 & a_2 + a_{23} r^2 \\ 0 & a_2 r^2 & a_0 + a_{33} r^2 \end{Vmatrix}, \tag{5·2}$$

where a_i, a_{ij} are constants and we have neglected terms of higher degree than the first in r^2. Corresponding expressions apply for β^i_j with a replaced by b.

From these expressions with (4·9) and (4·11) it follows that

$$\Delta_1 = \Delta - \omega^2 \delta_1 r^2 + O(r^4), \quad \Delta_2 = \Delta - \omega^2 \delta_2 r^2 + O(r^4), \tag{5·3}$$

where

$$\left.\begin{aligned} \Delta &= p^2 - \omega^2 a_0 b_0, \\ \delta_1 = a_0 b_{22} + a_1 b_0, &\quad \delta_2 = b_0 a_{22} + b_1 a_0. \end{aligned}\right\} \tag{5·4}$$

The coefficients (4·11) and (4·15) now take the forms

$$\left.\begin{aligned} &A_1 = A_{11} r^2 + O(r^4), && A_2 = A_{11} r + O(r^3), \\ &A_3 = -n^2 A_{11} + O(r^2), && \\ &B_2 = B_{21} r^3 + O(r^5), && B_3 = B_{31} r^2 + O(r^4), \\ &D_1 = -D_{11} r^2 + O(r^4), && D_2 = -D_{11} r + O(r^3), \\ &D_3 = n^2 D_{11} + O(r^2), && \\ &C_2 = -B_{21} r^3 + O(r^5), && C_3 = C_{31} r^2 + O(r^4), \end{aligned}\right\} \tag{5·5}$$

where

$$\left.\begin{aligned} &A_{11} = \omega a_0 \Delta^2, \quad D_{11} = \omega b_0 \Delta^2, \\ &B_{21} = \Delta \omega^2 [\Delta(a_0 b_2 - b_0 a_2) - np(\delta_1 - \delta_2)], \\ &B_{31} = 2\Delta\omega^2(\Delta a_0 b_2 - \delta_1 np), \quad C_{31} = 2\Delta\omega^2(\Delta a_2 b_0 - \delta_2 np). \end{aligned}\right\} \tag{5·6}$$

Consistent with (5·1), to obtain equations for k and l we may write

$$h_z = c_0 r^k, \quad e_z = d_0 r^l, \tag{5·7}$$

in (4·12) and (4·13) and, using (5·5), we obtain

$$\left.\begin{aligned} \mathrm{i}A_{11} d_0 (l^2 - n^2)\, r^l + c_0 (B_{21} k + B_{31})\, r^{k+2} = 0, \\ \mathrm{i}D_{11} c_0 (k^2 - n^2)\, r^k + d_0 (B_{21} l - C_{31})\, r^{l+2} = 0, \end{aligned}\right\} \tag{5·8}$$

c_0 and d_0 being complex constants. In (5·8) we assume that $B_{21} k + B_{31} \neq 0$ and $B_{21} l - C_{31} \neq 0$. If either of these quantities become zero, we should need to consider the effect of terms involving higher powers of r in the coefficients (5·5).

Since $e_z \equiv 0$ implies $h_z \equiv 0$, from (4·22), and $h_z \equiv 0$ implies that $e_z \equiv 0$, from (4·19), we consider solutions for which neither e_z nor h_z is identically zero. In (5·8) we therefore require $c_0 \neq 0$ and $d_0 \neq 0$. If l [illegible] we conclude from the first of (5·8), by letting $r \to 0$, that $c_0 = 0$. Similarly $d_0 = 0$ if $k > l+2$. It follows that $k - 2 \leqslant l \leqslant k+2$. If $l = k+1$ the first of (5·8) can be satisfied with $d_0 \neq 0$ as $r \to 0$, provided $l = \pm n$. However, the second of (5·8) would then require $k = \pm n$ which is impossible, or $c_0 = 0$, which we have excluded. The possibility $k = l+1$ must be ruled out on similar grounds.

If $l = k+2$, it follows from the second of (5·8) that $c_0 \neq 0$ provided $k = \pm n$. From the first of (5·8), we then obtain the relations

$$\left.\begin{aligned} 4iA_{11}(n+1)\,d_0+(B_{21}n+B_{31})\,c_0 &= 0 \quad (k = n \neq 0),\\ 4iA_{11}(n-1)\,d_0+(B_{21}n-B_{31})\,c_0 &= 0 \quad (k = -n \neq 0).\end{aligned}\right\} \tag{5·9}$$

When $k = l-2 = \pm n$, equation (4·22) becomes identical with the first of (5·8) as $r \to 0$.

Similarly if $k = l+2$, it follows from the first of (5·8) that $d_0 \neq 0$ provided $l = \pm n$. From the second of these equations we then obtain

$$\left.\begin{aligned} 4iD_{11}(n+1)\,c_0+(B_{21}n-C_{31})\,d_0 &= 0 \quad (l = n \neq 0),\\ 4iD_{11}(n-1)\,c_0+(B_{21}n+C_{31})\,d_0 &= 0 \quad (l = -n \neq 0),\end{aligned}\right\} \tag{5·10}$$

and, in this case, (4·19) becomes identical with the second of (5·8) as $r \to 0$.

We have thus indicated the existence of four independent series solutions of (4·12), (4·13) with the leading terms taking the forms (5·7). For two of these $k = l+2$, for the others $l = k+2$.

To indicate the behaviour at $r = 0$ of the functions defined in (4·17) and (4·20) we use the notation

$$\left.\begin{aligned} (R_1, R_2, R_3, R_4) &= (R_{|n|}, R_{|n|+2}, R_{-|n|}, R_{-|n|+2}),\\ (S_1, S_2, S_3, S_4) &= (S_{|n|+2}, S_{|n|}, S_{-|n|+2}, S_{-|n|}),\end{aligned}\right\} \tag{5·11}$$

the suffixes in the right-hand members of these equations being the values of k and l in the series solutions (5·1).

Introducing (5·11) into (4·23), we obtain

$$\left.\begin{aligned} h_z &= K_1R_{|n|}+K_2R_{|n|+2}+K_3R_{-|n|}+K_4R_{-|n|+2},\\ \text{and}\quad e_z &= K_1S_{|n|+2}+K_2S_{|n|}+K_3S_{-|n|+2}+K_4S_{-|n|}.\end{aligned}\right\} \tag{5·12}$$

If no singularity in either e_z and h_z can exist at $r = 0$, we must have $K_3 = K_4 = 0$, and (5·12) then yields

$$\left.\begin{aligned} h_z &= K_1R_{|n|}+K_2R_{|n|+2},\\ \text{and}\quad e_z &= K_1S_{|n|+2}+K_2S_{|n|}.\end{aligned}\right\} \tag{5·13}$$

Equations (5·8) can also be satisfied with non-zero values of c_0 and d_0 as $r \to 0$ if $k = l = \pm n$. To determine whether the series solutions for e_z and h_z in each of these cases can be mutually consistent we should need to consider the higher-order terms in the expansions (5·1) and (5·8). We may, however, exclude these further possible solutions since the equations for e_z and h_z derived from (4·12) and (4·13) are each linear and of the fourth order. This implies that any further solutions of these equations must be expressible as a linear combination of those defined in (5·11).

6. Nature of the transmitted wave

In the application of formulae of § 4, the quantities e_r, e_θ, h_r and h_θ are obtained by introducing into (4·8) and (4·10) the solutions for e_z and h_z appropriate to the problem being examined. These solutions are given by (4·23), (5·12) or (5·13). The components of $\bar{\mathbf{e}}$ and $\bar{\mathbf{h}}$ are derived from (4·5) and the field vectors $\mathbf{E}$ and $\mathbf{H}$ are then given by (2·6). The constants

K_μ and p are determined from the boundary conditions and in general p emerges as the result of an eigenvalue problem. The manner in which p is determined is illustrated by the problems of the coaxial and circular wave guides examined in §§7 and 8.

We consider now the effect of combining the two solutions for which $n = m$ and $n = -m$ where m is a positive integer. For these two values of n we obtain from (4·14) and (4·15) different expressions for A_3, B_2, B_3, C_2, C_3 and D_3. Equations (4·12) and (4·13) then yield different solutions for h_z and e_z and, in general, the values of p determined by the boundary conditions will also differ. We distinguish quantities associated with the solutions for which $n = m$ and $n = -m$ by the suffixes 1 and 2, respectively.

We therefore superpose the two solutions

$$\mathbf{E}_1 = \mathscr{R}\bar{\mathbf{e}}_1 \exp(\mathrm{i}\omega t), \qquad \mathbf{E}_2 = \mathscr{R}\bar{\mathbf{e}}_2 \exp(\mathrm{i}\omega t), \tag{6·1}$$

where

$$\bar{\mathbf{e}}_1 = \mathbf{e}_1^* \exp[\mathrm{i}(m\theta - p_1 z)], \quad \bar{\mathbf{e}}_2 = \mathbf{e}_2^* \exp[-\mathrm{i}(m\theta + p_2 z)] \tag{6·2}$$

correspond to (4·2), $\mathbf{e}_1^*$, $\mathbf{e}_2^*$ being functions of r only.

In general, p_1 and p_2 may be complex and we write

$$p_1 = p_1^+ + \mathrm{i}p_1^-, \quad p_2 = p_2^+ + \mathrm{i}p_2^-. \tag{6·3}$$

From (6·1) and (6·2) we see that the amplitude of $\mathbf{E}_1$ increases in the direction of propagation if p_1^+ and p_1^- are either both positive or both negative. If this occurs for any value of ω the system becomes unstable and we shall assume that this situation does not arise. Similarly we exclude the possibility that p_2^+ and p_2^- are either both positive or both negative.

To simplify subsequent expressions we write

$$\left.\begin{aligned}\mathbf{e}_1 &= \mathbf{e}_1^* \exp(p_1^- z) = \mathbf{e}_1^+ + \mathrm{i}\mathbf{e}_1^-,\\ \mathbf{e}_2 &= \mathbf{e}_2^* \exp(p_2^- z) = \mathbf{e}_2^+ + \mathrm{i}\mathbf{e}_2^-,\end{aligned}\right\} \tag{6·4}$$

and

$$\left.\begin{aligned}\phi &= \tfrac{1}{2}(p_1^+ + p_2^+)\,z - \omega t,\\ \delta &= \tfrac{1}{2}(p_1^+ - p_2^+)/m, \quad \bar{\theta} = \theta - \delta z.\end{aligned}\right\} \tag{6·5}$$

In (6·4) $\mathbf{e}_1^+$, $\mathbf{e}_1^-$, $\mathbf{e}_2^+$, $\mathbf{e}_2^-$ are real vectors.

The composite wave $\mathbf{E}$ obtained from $\mathbf{E}_1$ and $\mathbf{E}_2$ may then be written as

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2\\ &= \mathscr{R}[\mathbf{e}_1 \exp\{\mathrm{i}(m\theta - p_1^+ z + \omega t)\} + \mathbf{e}_2 \exp\{-\mathrm{i}(m\theta + p_2^+ z - \omega t)\}]\\ &= \mathscr{R}[\mathbf{e}_1 \exp\{\mathrm{i}(m\bar{\theta} - \phi)\} + \mathbf{e}_2 \exp\{-\mathrm{i}(m\bar{\theta} + \phi)\}].\end{aligned} \tag{6·6}$$

We therefore have

$$\mathbf{E} = \mathscr{R}[(\mathbf{E}^+ + \mathrm{i}\mathbf{E}^-) \exp(-\mathrm{i}\phi)] = \mathbf{E}^+ \cos\phi + \mathbf{E}^- \sin\phi, \tag{6·7}$$

where

$$\left.\begin{aligned}\mathbf{E}^+ &= [(\mathbf{e}_1^+ + \mathbf{e}_2^+)\cos m\bar{\theta} - (\mathbf{e}_1^- - \mathbf{e}_2^-)\sin m\bar{\theta}],\\ \mathbf{E}^- &= [(\mathbf{e}_1^- + \mathbf{e}_2^-)\cos m\bar{\theta} + (\mathbf{e}_1^+ - \mathbf{e}_2^+)\sin m\bar{\theta}].\end{aligned}\right\} \tag{6·8}$$

Equations (6·7) and (6·8) describe an elliptically polarized wave, the vector $\mathbf{E}$ executing an ellipse as ϕ (or t) varies. The semi-axes Λ_1, Λ_2 of this ellipse are given by

$$\Lambda_1^2, \Lambda_2^2 = \tfrac{1}{2}\{(\mathbf{E}^+.\mathbf{E}^+ + \mathbf{E}^-.\mathbf{E}^-) \pm [(\mathbf{E}^+.\mathbf{E}^+ - \mathbf{E}^-.\mathbf{E}^-)^2 + 4(\mathbf{E}^+.\mathbf{E}^-)^2]^{\frac{1}{2}}\} \tag{6·9}$$

these quantities, and the orientation of the ellipse, varying, in general, with r and $\bar{\theta}$.

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If there is no attenuation, so that $p_1^- = p_2^- = 0$, $\mathbf{e}_1^+$, $\mathbf{e}_1^-$, $\mathbf{e}_2^+$ and $\mathbf{e}_1^-$ are functions only of r. The vectors $\mathbf{E}^+$ and $\mathbf{E}^-$ defined by (6·8) then involve θ and z only in the combination $\bar{\theta} = \theta - \delta z$ and the wave is transmitted without distortion, but in such a manner that the field pattern rotates through an angle $\delta = (p_1^+ - p_2^+)/(2m)$ per unit distance travelled in the z-direction.

If each of the component waves $\mathbf{E}_1$ and $\mathbf{E}_2$ suffers an equal amount of attenuation during transmission we have

$$p_1^- = p_2^- = -p^- \quad \text{(say)}$$

and $\mathbf{E}^+$ and $\mathbf{E}^-$ become functions of r and $\bar{\theta}$ multiplied by the same exponential factor $\exp(-p^- z)$. The field again rotates and is then transmitted without distortion, the shape and orientation of the polarization ellipse being the same for given values of r and $\bar{\theta}$ throughout the tube.

If $p_1^- \neq p_2^-$, $\mathbf{E}_1$ and $\mathbf{E}_2$ undergo different amounts of attenuation. The ratio Λ_1/Λ_2 is then a function of z and the wave is distorted as it travels along the tube.

If the direction of propagation is reversed, this direction bears the same relation as the original to the frame of reference if we reverse the sense in which θ and z are measured, that is, replace θ, z by $-\theta$, $-z$ in (4·1) and (4·2). The sign of τ is unaffected by this change of co-ordinates. In subsequent equations n and p are then replaced by $-n$ and $-p$, respectively. This leaves unchanged the coefficients (4·14) and (4·15) and the differential equations (4·12) and (4·13). The values of p and hence also the rate of rotation relative to the new co-ordinate system are then unchanged. The wave pattern therefore rotates back along the path previously traversed and the system is reciprocal. This property may also be inferred immediately from the fact that the tensors $g_{ir}\alpha_j^r$, $g_{ir}\beta_j^r$ are symmetric (Katz 1959).

7. The coaxial waveguide

We consider propagation of an electromagnetic wave along a circular tube subjected to the deformation (3·5) and bounded in the deformed state by the cylindrical surfaces $r = r_1$, $r = r_2$ $(r_1 > r_2)$. We assume that these boundaries are perfectly conducting so that here the tangential components of $\mathbf{E}$ vanish. This implies that at $r = r_1$, $r = r_2$

$$e_\theta = e_z = 0. \tag{7·1}$$

From (4·10) we see that these conditions may be replaced by

$$e_z = 0, \quad \mathrm{d}h_z/\mathrm{d}r = 0 \tag{7·2}$$

at $r = r_1$, $r = r_2$.

The general solutions for e_z and h_z take the forms (4·23). From the boundary conditions (7·2) we then have

$$\left.\begin{aligned} &\sum_{\mu=1}^{4} K_\mu R'_\mu(r_1) = 0, \quad \sum_{\mu=1}^{4} K_\mu R'_\mu(r_2) = 0,\\ &\sum_{\mu=1}^{4} K_\mu S_\mu(r_1) = 0, \quad \sum_{\mu=1}^{4} K_\mu S_\mu(r_2) = 0, \end{aligned}\right\} \tag{7·3}$$

primes denoting differentiation with respect to r. These equations for K_μ are consistent if

$$\begin{vmatrix} R'_1(r_1) & R'_2(r_1) & R'_3(r_1) & R'_4(r_1) \\ R'_1(r_2) & R'_2(r_2) & R'_3(r_2) & R'_4(r_2) \\ S_1(r_1) & S_2(r_1) & S_3(r_1) & S_4(r_1) \\ S_1(r_2) & S_2(r_2) & S_3(r_2) & S_4(r_2) \end{vmatrix} = 0. \tag{7·4}$$

Remembering (4·17) and (4·20) we see that this represents an equation for p when τ, ω, n and the dimensions of the tube are prescribed. The two values $n = m$, $n = -m$ where m is a positive integer, yield, in general, two values for p and the combination of these two solutions yields, as in §6, a wave which rotates during propagation.

8. The circular waveguide

A corresponding analysis may be applied to the propagation of an electromagnetic wave along a solid circular rod subjected to torsion. In this case, the analysis of §5 is applicable and e_z and h_z are given by (5·13). We again assume that the cylindrical surface $r = r_1$ is perfectly conducting, so that on $r = r_1$

$$e_z = 0, \quad \mathrm{d}h_z/\mathrm{d}r = 0. \tag{8·1}$$

Introducing (5·13) into (8·1), we obtain

$$\left.\begin{aligned} K_1 R'_{|n|}(r_1) + K_2 R'_{|n|+2}(r_1) &= 0, \\ K_1 S_{|n|+2}(r_1) + K_2 S_{|n|}(r_1) &= 0. \end{aligned}\right\} \tag{8·2}$$

From these we obtain the secular equation

$$\begin{vmatrix} R'_{|n|}(r_1) & R'_{|n|+2}(r_1) \\ S_{|n|+2}(r_1) & S_{|n|}(r_1) \end{vmatrix} = 0 \tag{8·3}$$

for p.

The two solutions for which $n = +m$, $n = -m$ when combined, again yield a rotating wave as discussed in §6.

SMALL TORSION

9. General theory

When the angle of torsion τ is small, the electromagnetic field may be obtained by means of a perturbation method from that existing in the undeformed material. If $r = r_1$ is the outer curved surface of the deformed rod or tube we assume that τ is sufficiently small so that $\tau r_1 \ll 1$.

Since r is an even function of τ we then have*

$$r' = 1 + O(\tau^2 r^2), \tag{9·1}$$

and from (3·9), the coefficients α^i_j take the forms

$$\|\alpha^i_j\| = \begin{Vmatrix} a_0 + O(\tau^2 r^2) & 0 & 0 \\ 0 & a_0 + O(\tau^2 r^2) & \tau[a_1 + O(\tau^2 r^2)] \\ 0 & \tau r^2[a_1 + O(\tau^2 r^2)] & a_0 + O(\tau^2 r^2) \end{Vmatrix}, \tag{9·2}$$

while for β^i_j we have corresponding expressions with a_0, a_1 replaced by b_0, b_1, respectively. The constants a_0, b_0, a_1 and b_1 are the values of α_0, β_0, α_1 and β_1 respectively at $\tau = 0$ and a_0 and b_0 are therefore the dielectric constant and magnetic permeability respectively for the undeformed material.

Since p depends upon τ through equations of the type (7·4) or (8·3), we write

$$p = {}^0p + \tau\, {}^1p + O(\tau^2 r_1^2), \tag{9·3}$$

* The theory of §§9, 10 and 11 is not restricted to incompressible materials.

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where ${}^0p$ and ${}^1p$ are independent of τ. To the first order in τr_1, we then obtain from (4·9) and (4·11)

$$\Delta_1 = \Delta_2 = -K^2 + 2\,{}^0p\,{}^1p\tau, \tag{9·4}$$

where

$$K^2 = \omega^2 a_0 b_0 - {}^0p^2. \tag{9·5}$$

The expressions (4·14) and (4·15) then yield

$$\left.\begin{aligned}
A_1 &= \omega\Delta_1^2 a_0 r^2 + O(\tau^2 r^2), \quad A_2 = A_1/r + O(\tau^2 r^2),\\
A_3 &= \omega\Delta_1^2\{(K^2 - n^2/r^2)\,a_0 + 2\,{}^0p(na_1 - a_0\,{}^1p)\,\tau\}\,r^2 + O(\tau^2 r^2),\\
B_2 &= \omega^2\Delta_1^2(a_0 b_1 - b_0 a_1)\,\tau r^3 + O(\tau^2 r^2),\\
B_3 &= 2\omega^2\Delta_1^2 a_0 b_1 \tau r^2 + O(\tau^2 r^2),\\
D_1 &= -\omega\Delta_1^2 b_0 r^2 + O(\tau^2 r^2), \quad D_2 = D_1/r + O(\tau^2 r^2),\\
D_3 &= -\omega\Delta_1^2\{(K^2 - n^2/r^2)\,b_0 + 2\,{}^0p(nb_1 - b_0\,{}^1p)\,\tau\}\,r^2 + O(\tau^2 r^2),\\
C_2 &= -B_2 + O(\tau^2 r^2),\\
C_3 &= 2\omega^2\,\Delta_1^2 a_1 b_0 \tau r^2 + O(\tau^2 r^2).
\end{aligned}\right\} \tag{9·6}$$

We assume that to the first order in τr, the vectors $\mathbf{e}$ and $\mathbf{h}$ take the forms

$$\mathbf{e} = {}^0\mathbf{e} + \tau\,{}^1\mathbf{e}, \quad \mathbf{h} = {}^0\mathbf{h} + \tau\,{}^1\mathbf{h}, \tag{9·7}$$

with corresponding expressions for the physical components defined in (4·5). Introducing these expansions with (9·6) into (4·12) and (4·13) and equating to zero the terms independent of τ, we obtain the usual Bessel equations for ${}^0e_z$ and ${}^0h_z$. Thus

$$\left.\begin{aligned}
\frac{\mathrm{d}^2({}^0e_z)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r} + \left(K^2 - \frac{n^2}{r^2}\right){}^0e_z &= 0,\\
\frac{\mathrm{d}^2({}^0h_z)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r} + \left(K^2 - \frac{n^2}{r^2}\right){}^0h_z &= 0.
\end{aligned}\right\} \tag{9·8}$$

The coefficients of τ in (4·12) and (4·13) yield similarly

$$\left.\begin{aligned}
&a_0\left\{\frac{\mathrm{d}^2({}^1e_z)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r} + \left(K^2 - \frac{n^2}{r^2}\right){}^1e_z\right\}\\
&\qquad = 2\,{}^0p({}^1pa_0 - na_1)\,{}^0e_z + \mathrm{i}\omega\{(a_0 b_1 - b_0 a_1)\,r\,(\mathrm{d}\,{}^0h_z/\mathrm{d}r) + 2a_0 b_1\,{}^0h_z\},\\
&b_0\left\{\frac{\mathrm{d}^2({}^1h_z)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r} + \left(K^2 - \frac{n^2}{r^2}\right){}^1h_z\right\}\\
&\qquad = 2\,{}^0p({}^1pb_0 - nb_1)\,{}^0h_z + \mathrm{i}\omega\{(a_0 b_1 - b_0 a_1)\,r\,(\mathrm{d}\,{}^0e_z/\mathrm{d}r) - 2a_1 b_0\,{}^0e_z\}.
\end{aligned}\right\} \tag{9·9}$$

Expressions for the remaining field components are derived by introducing (9·3), (9·4) and the expansions for e_z and h_z into (4·8) and (4·10). We obtain from the terms independent of τ

$$\left.\begin{aligned}
{}^0e_r &= -\{\mathrm{i}\,{}^0pr(\mathrm{d}\,{}^0e_z/\mathrm{d}r) - n\omega b_0\,{}^0h_z\}/(K^2 r),\\
{}^0e_\theta &= \{n\,{}^0p\,{}^0e_z + \mathrm{i}\omega b_0 r(\mathrm{d}\,{}^0h_z/\mathrm{d}r)\}/(K^2 r),\\
{}^0h_r &= -\{\mathrm{i}\,{}^0pr(\mathrm{d}\,{}^0h_z/\mathrm{d}r) + n\omega a_0\,{}^0e_z\}/(K^2 r),\\
{}^0h_\theta &= \{n\,{}^0p\,{}^0h_z - \mathrm{i}\omega a_0 r(\mathrm{d}\,{}^0e_z/\mathrm{d}r)\}/(K^2 r),
\end{aligned}\right\} \tag{9·10}$$

and similarly from the terms linear in τ we have

$$\left.\begin{aligned}
{}^1e_r &= -\frac{1}{K^2r}\left\{\mathrm{i}\,{}^0pr\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r} - n\omega b_0\,{}^1h_z + \mathrm{i}rq_1\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r} + \omega({}^0pb_1r^2 - nb_0q_2)\,{}^0h_z\right\},\\
{}^1e_\theta &= \frac{1}{K^2r}\left\{n\,{}^0p\,{}^1e_z + \mathrm{i}\omega b_0 r\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r} + (nq_1 - \omega^2a_1b_0r^2)\,{}^0e_z + \mathrm{i}\omega b_0q_2r\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r}\right\},\\
{}^1h_r &= -\frac{1}{K^2r}\left\{\mathrm{i}\,{}^0pr\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r} + n\omega a_0\,{}^1e_z + \mathrm{i}rq_1\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r} - \omega({}^0pa_1r^2 - na_0q_2)\,{}^0e_z\right\},\\
{}^1h_\theta &= \frac{1}{K^2r}\left\{n\,{}^0p\,{}^1h_z - \mathrm{i}\omega a_0 r\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r} + (nq_1 - \omega^2a_0b_1r^2)\,{}^0h_z - \mathrm{i}\omega a_0q_2r\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r}\right\},
\end{aligned}\right\} \tag{9·11}$$

where $$q_1 = \left(\frac{\omega^2a_0b_0 + {}^0p^2}{\omega^2a_0b_0 - {}^0p^2}\right){}^1p, \quad q_2 = \frac{2\,{}^0p\,{}^1p}{K^2}. \tag{9·12}$$

10. Circular waveguide: transverse magnetic wave

We examine again the problem discussed in §8 of a wave propagated along a solid circular rod, but restrict attention to the case where the amount of torsion τ is small. Assuming that the outer surface $r = r_1$ is perfectly conducting, and using the notation of §9, the boundary conditions (8·1) yield at $r = r_1$

$${}^0e_z = 0, \quad \mathrm{d}\,{}^0h_z/\mathrm{d}r = 0, \tag{10·1}$$

$${}^1e_z = 0, \quad \mathrm{d}\,{}^1h_z/\mathrm{d}r = 0. \tag{10·2}$$

For the undeformed rod, when $\tau = 0$, propagation is governed by equation (9·8) and we may, following the usual procedure (see, for example, Lamont 1946), distinguish between transverse electric or H-waves in which ${}^0e_z \equiv 0$ and transverse magnetic or E-waves in which ${}^0h_z \equiv 0$. Any other wave propagated along the rod may be expressed as a linear combination of waves of these two types. For the twisted rod we may examine modes of propagation based upon these two kinds of wave in the undeformed material.

Considering first the perturbation of a transverse magnetic wave, we obtain from (9·8), when $n = m$ $(m > 0)$ the solution*

$${}^0e_z = {}^0e_z^+ = AJ_m(Kr), \quad {}^0h_z \equiv 0. \tag{10·3}$$

where A is a real arbitrary constant and $J_m(Kr)$ is the Bessel function of order m of the first kind. The Bessel function of the second kind $Y_m(Kr)$ is excluded from the solution for ${}^0e_z$ since ${}^0e_z$ is finite at $r = 0$. To satisfy the boundary conditions (10·1), K must be a root of the equation

$$J_m(Kr_1) = 0, \tag{10·4}$$

and in view of (9·5) this may be regarded as the secular equation for ${}^0p$.

* We use the superscripts + and − to denote the real and imaginary parts respectively of a complex quantity. We assume for the present that in (2·2) α_i, β_i, α_j^i and β_j^i are real and therefore that the constants a_0, a_1, b_0, b_1 defined in §9 are also real. Equations (9·8) are the equations of classical waveguide theory and for an unattenuated wave ${}^0p$ and hence also K and $J_m(Kr)$ are real. We may therefore choose A to be real; the assumption of a complex value leads to no essentially new results. This part of the analysis may, of course, be carried through formally with complex values for ${}^0p$, K and A, but in either case for the perturbation component we may choose ${}^1p$ to have the real value (10·5). Similar remarks apply to the remaining solutions of §§10, 11, 13 and 14.

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Expressions for ${}^1e_z$, ${}^1h_z$ may be derived from (9·9) by introducing (10·3) and solving the resulting equations by the method of variation of parameters. If we choose

$$ {}^1p = ma_1/a_0 = {}^1p_1 \quad \text{(say)} \tag{10·5}$$

and write $$f_h(r) = (\omega/b_0)\{(a_0b_1 - a_1b_0)\,KrJ'_m(Kr) - 2a_1b_0J_m(Kr)\}, \tag{10·6}$$

the first of (9·9) may be satisfied with ${}^1e_z \equiv 0$, while the second equation yields

$$ {}^1h_z = \mathrm{i}\,{}^1h_z^-, \quad {}^1h_z^- = AF_h(r), \tag{10·7}$$

where

$$F_h(r) = \tfrac{1}{2}\pi\left\{Y_m(Kr)\int_0^r J_m(Ku)f_h(u)\,u\,\mathrm{d}u - J_m(Kr)\int_0^r Y_m(Ku)f_h(u)\,u\,\mathrm{d}u + MJ_m(Kr)\right\}, \tag{10·8}$$

$J'_m(x) = \mathrm{d}J_m(x)/\mathrm{d}x$ and M is a real constant. In deriving (10·8) we have used the relation (Watson 1944, §3·63)

$$[J_m(Kr)\,Y'_m(Kr) - J'_m(Kr)\,Y_m(Kr)]^{-1} = \tfrac{1}{2}\pi Kr. \tag{10·9}$$

We notice that as $r \to 0$

$$\left.\begin{aligned} J_m(Kr) &= O(r^m), \quad Y_m(Kr) = O(r^{-m}),\\ f_h(r) &= O(r^m), \end{aligned}\right\} \tag{10·10}$$

so that all terms in (10·8) are finite at $r = 0$. The boundary conditions (10·2) may be satisfied by choosing M so that

$$MJ'_m(Kr_1) + Y'_m(Kr_1)\int_0^{r_1} J_m(Ku)f_h(u)\,u\,\mathrm{d}u - J'_m(Kr_1)\int_0^{r_1} Y_m(Ku)f_h(u)\,u\,\mathrm{d}u = 0. \tag{10·11}$$

The foregoing solution with $n = m$ yields expressions for the vectors $\mathbf{e}_1$ and $\mathbf{E}_1$ of §6 and the corresponding magnetic vectors $\mathbf{h}_1$ and $\mathbf{H}_1$. The physical components of $\mathbf{e}_1$ and $\mathbf{h}_1$ may be evaluated by introducing (10·3) and (10·7) into (9·10) and (9·11) and making use of (9·7). If we denote these components by $(e_{1r}, e_{1\theta}, e_{1z})$, $(h_{1r}, h_{1\theta}, h_{1z})$ respectively we obtain

$$\left.\begin{aligned} (e_{1r}, e_{1\theta}, e_{1z}) &= [\mathrm{i}({}^0e_r^- + \tau\,{}^1e_r^-),\quad {}^0e_\theta^+ + \tau\,{}^1e_\theta^+,\quad {}^0e_z^+],\\ (h_{1r}, h_{1\theta}, h_{1z}) &= [{}^0h_r^+ + \tau\,{}^1h_r^+,\quad \mathrm{i}({}^0h_\theta^- + \tau\,{}^1h_\theta^-),\quad \mathrm{i}\tau\,{}^1h_z^-], \end{aligned}\right\} \tag{10·12}$$

where

$$\left.\begin{aligned} {}^0e_r^- &= -(A\,{}^0p/K)\,J'_m(Kr), \quad {}^0e_\theta^+ = [Am\,{}^0p/(K^2r)]\,J_m(Kr), \quad {}^0e_z^+ = AJ_m(Kr),\\ {}^0h_r^+ &= -[Am\omega a_0/(K^2r)]\,J_m(Kr), \quad {}^0h_\theta^- = -(A\omega a_0/K)\,J'_m(Kr), \end{aligned}\right\} \tag{10·13}$$

and

$$\left.\begin{aligned} {}^1e_r^- &= A\{m\omega b_0F_h(r) - rq_1KJ'_m(Kr)\}/(K^2r),\\ {}^1e_\theta^+ &= -A\{\omega b_0rF'_h(r) - (mq_1 - \omega^2a_1b_0r^2)\,J_m(Kr)\}/(K^2r),\\ {}^1h_r^+ &= A\{{}^0prF'_h(r) + \omega({}^0pa_1r^2 - a_0mq_2)\,J_m(Kr)\}/(K^2r),\\ {}^1h_\theta^- &= A\{m\,{}^0pF_h(r) - \omega a_0q_2KrJ'_m(Kr)\}/(K^2r),\\ {}^1h_z^- &= AF_h(r). \end{aligned}\right\} \tag{10·14}$$

In these equations $F_h(r)$ is given by (10·8) and ${}^0p$ by (9·5) and (10·4). Also, from (10·5) and (9·12) we have

$$q_1 = m\left(\frac{\omega^2a_0b_0 + {}^0p^2}{\omega^2a_0b_0 - {}^0p^2}\right)\frac{a_1}{a_0}, \quad q_2 = \frac{2ma_1\,{}^0p}{a_0K^2}. \tag{10·15}$$

When $n = -m$ $(m > 0)$ equations (10·1) to (10·4) and (10·6) to (10·11) are unchanged apart from the replacement of A by a different arbitrary constant A' which we assume to be real. Equations (9·5) and (10·4) continue to determine ${}^0p$ but (10·5) is now replaced by

$$ {}^1p = -ma_1/a_0 = {}^1p_2 \quad \text{(say)}. \tag{10·16}$$

As before, we may now derive expressions for the vectors $\mathbf{e}_2$ and $\mathbf{E}_2$ of §6 and the corresponding magnetic vectors $\mathbf{h}_2$ and $\mathbf{H}_2$. Distinguishing the physical components of $\mathbf{e}_2$ and $\mathbf{h}_2$ by the additional suffix 2, we then have

$$\left.\begin{aligned}(e_{2r}, e_{2\theta}, e_{2z}) &= [\mathrm{i}l({}^0e_r^- - \tau\,{}^1e_r^-), \quad -l({}^0e_\theta^+ - \tau\,{}^1e_\theta^+), \quad l\,{}^0e_z^+],\\ (h_{2r}, h_{2\theta}, h_{2z}) &= [-l({}^0h_r^+ - \tau\,{}^1h_r^+), \quad \mathrm{i}l({}^0h_\theta^- - \tau\,{}^1h_\theta^-), \quad \mathrm{i}l\tau\,{}^1h_z^-],\end{aligned}\right\} \tag{10·17}$$

where
$$ l = A'/A, \tag{10·18}$$

and ${}^0e_r^-, \ldots, {}^1h_z^-$ are given by (10·13) and (10·14).

In the notation of §6, we have, from (9·3)

$$ p_1 = p_1^+ = {}^0p + \tau\,{}^1p_1, \quad p_2 = p_2^+ = {}^0p + \tau\,{}^1p_2, \tag{10·19}$$

and equations (6·5), (10·5) and (10·16) yield

$$ \phi = {}^0pz - \omega t, \quad \delta = \tau a_1/a_0, \tag{10·20}$$

and $\bar{\theta} = \theta - \tau a_1 z/a_0$.

If we denote the physical components of $\mathbf{E}$ by (E_r, E_θ, E_z) we obtain from (6·7), (6·8), (10·12) and (10·17)

$$\left.\begin{aligned}E_r &= l({}^0e_r^- - \tau\,{}^1e_r^-)\sin(m\bar{\theta}+\phi) - ({}^0e_r^- + \tau\,{}^1e_r^-)\sin(m\bar{\theta}-\phi),\\ E_\theta &= -l({}^0e_\theta^+ - \tau\,{}^1e_\theta^+)\cos(m\bar{\theta}+\phi) + ({}^0e_\theta^+ + \tau\,{}^1e_\theta^+)\cos(m\bar{\theta}-\phi),\\ E_z &= \{l\cos(m\bar{\theta}+\phi) + \cos(m\bar{\theta}-\phi)\}\,{}^0e_z^+.\end{aligned}\right\} \tag{10·21}$$

Similarly, from (10·12), (10·17) and equations analogous to (6·7) and (6·8) we obtain for the physical components (H_r, H_θ, H_z) of $\mathbf{H}$

$$\left.\begin{aligned}H_r &= -l({}^0h_r^+ - \tau\,{}^1h_r^+)\cos(m\bar{\theta}+\phi) + ({}^0h_r^+ + \tau\,{}^1h_r^+)\cos(m\bar{\theta}-\phi),\\ H_\theta &= l({}^0h_\theta^- - \tau\,{}^1h_\theta^-)\sin(m\bar{\theta}+\phi) - ({}^0h_\theta^- + \tau\,{}^1h_\theta^-)\sin(m\bar{\theta}-\phi),\\ H_z &= \{l\sin(m\bar{\theta}+\phi) - \sin(m\bar{\theta}-\phi)\}\,\tau\,{}^1h_z^-.\end{aligned}\right\} \tag{10·22}$$

Remembering the discussion of §6, we see that equations (10·21) and (10·22) describe a wave which is elliptically polarized at each point. Since p_0 and p_1 are real, the wave is propagated without distortion or attenuation, but each of the fields is rotated through an angle $\tau a_1/a_0$ per unit length travelled along the guide. From (2·1), (2·2) and (9·2) we see that a_0 may be identified with the dielectric constant of the undeformed material and is therefore positive. From (10·20) and (3·5) it then follows that if a_1 is negative the fields rotate in a direction opposite to that in which the tube is twisted, while the directions of torsion and rotation of the fields are the same if a_1 is positive. In addition to this rotation, a small longitudinal component of magnetic field is introduced by the torsion.

Complex values for A and l affect only the relative magnitudes Λ_1, Λ_2 and the directions of the axes of the polarization ellipse at each point. The values of p_0 and p_1 and hence also the rate of rotation of the wave are unaffected.

If $l = 1$ equations (10·21) and (10·22) reduce to

$$\left.\begin{aligned} E_r &= 2\{{}^0e_r^- \cos m\theta \sin\phi - \tau\,{}^1e_r^- \sin m\theta \cos\phi\}, \\ E_\theta &= 2\{{}^0e_\theta^+ \sin m\theta \sin\phi + \tau\,{}^1e_\theta^+ \cos m\theta \cos\phi\}, \\ E_z &= 2\,{}^0e_z^+ \cos m\theta \cos\phi, \\ H_r &= 2\{{}^0h_r^+ \sin m\theta \sin\phi + \tau\,{}^1h_r^+ \cos m\theta \cos\phi\}, \\ H_\theta &= 2\{{}^0h_\theta^- \cos m\theta \sin\phi - \tau\,{}^1h_\theta^- \sin m\theta \cos\phi\}, \\ H_z &= 2\tau\,{}^1h_z^- \cos m\theta \sin\phi. \end{aligned}\right\} \tag{10·23}$$

11. Circular waveguide: transverse electric wave

For the alternative mode of propagation, we choose the solution

$${}^0e_z \equiv 0, \quad {}^0h_z = {}^0h_z^+ = AJ_m(Kr) \quad (m = |n|) \tag{11·1}$$

of (9·8), where A is again a real arbitrary constant. The boundary conditions (10·1) are satisfied provided K is a root of

$$J_m'(Kr_1) = 0. \tag{11·2}$$

If we choose

$$\left.\begin{aligned} {}^1p &= mb_1/b_0 = {}^1p_1 \quad (n = m > 0), \\ \text{or} \quad {}^1p &= -mb_1/b_0 = {}^1p_2 \quad (n = -m < 0), \end{aligned}\right\} \tag{11·3}$$

and write
$$f_e(r) = (\omega/a_0)\{(a_0b_1 - a_1b_0)\,KrJ_m'(Kr) + 2a_0b_1J_m(Kr)\}, \tag{11·4}$$

the second of (9·9) may be satisfied with ${}^1h_z = 0$ while the first equation yields

$${}^1e_z = \mathrm{i}\,{}^1e_z^-, \quad {}^1e_z^- = AF_e(r), \tag{11·5}$$

where

$$F_e(r) = \tfrac{1}{2}\pi\left\{Y_m(Kr)\int_0^r J_m(Ku)f_e(u)\,u\,\mathrm{d}u - J_m(Kr)\int_0^r Y_m(Ku)f_e(u)\,u\,\mathrm{d}u + M'J_m(Kr)\right\}. \tag{11·6}$$

The function $F_e(r)$, like the corresponding function $F_h(r)$ defined by (10·8), is finite at $r = 0$. The boundary conditions (10·2) are satisfied by choosing M' so that

$$M'J_m(Kr_1) + Y_m(Kr_1)\int_0^{r_1} J_m(Ku)f_e(u)\,u\,\mathrm{d}u - J_m(Kr_1)\int_0^{r_1} Y_m(Ku)f_e(u)\,u\,\mathrm{d}u = 0. \tag{11·7}$$

The components of $\mathbf{e}$ and $\mathbf{h}$ are evaluated by combining (11·5) and (11·1) with (9·10), (9·11) and (9·7). As before we write $\mathbf{e}, \mathbf{h} = \mathbf{e}_1, \mathbf{h}_1$ when $n = m$ and $\mathbf{e}, \mathbf{h} = \mathbf{e}_2, \mathbf{h}_2$ when $n = -m$ with a corresponding notation for the components of these vectors. For the components of $\mathbf{e}_1, \mathbf{h}_1$ we then have

$$\left.\begin{aligned} (e_{1r}, e_{1\theta}, e_{1z}) &= [{}^0e_r^+ + \tau\,{}^1e_r^+, \quad \mathrm{i}({}^0e_\theta^- + \tau\,{}^1e_\theta^-), \quad \mathrm{i}\tau\,{}^1e_z^-], \\ (h_{1r}, h_{1\theta}, h_{1z}) &= [\mathrm{i}({}^0h_r^- + \tau\,{}^1h_r^-), \quad {}^0h_\theta^+ + \tau\,{}^1h_\theta^+, \quad {}^0h_z^+], \end{aligned}\right\} \tag{11·8}$$

while the components of $\mathbf{e}_2, \mathbf{h}_2$ take the forms

$$\left.\begin{aligned} (e_{2r}, e_{2\theta}, e_{2z}) &= [-l({}^0e_r^+ - \tau\,{}^1e_r^+), \quad \mathrm{i}l({}^0e_\theta^- - \tau\,{}^1e_\theta^-), \quad \mathrm{i}l\tau\,{}^1e_z^-], \\ (h_{2r}, h_{2\theta}, h_{2z}) &= [\mathrm{i}l({}^0h_r^- - \tau\,{}^1h_r^-), \quad -l({}^0h_\theta^+ - \tau\,{}^1h_\theta^+), \quad l\,{}^0h_z^+], \end{aligned}\right\} \tag{11·9}$$

where l is an arbitrary constant. In (11·8) and (11·9) the quantities ${}^0e_r^+$, ${}^0e_\theta^-$, ..., ${}^1h_\theta^+$ are given by

$$\left.\begin{aligned} {}^0e_r^+ &= [Am\omega b_0/(K^2r)]\,J_m(Kr), \quad {}^0e_\theta^- = (A\omega b_0/K)\,J_m'(Kr),\\ {}^0h_r^- &= -(A\,{}^0p/K)\,J_m'(Kr), \quad {}^0h_\theta^+ = [Am\,{}^0p/(K^2r)]\,J_m(Kr),\\ {}^0h_z^+ &= AJ_m(Kr). \end{aligned}\right\} \tag{11·10}$$

$$\left.\begin{aligned} {}^1e_r^+ &= A\{{}^0prF_e'(r)-\omega({}^0pb_1r^2-mb_0q_2)\,J_m(Kr)\}/(K^2r),\\ {}^1e_\theta^- &= A\{m\,{}^0pF_e(r)+\omega b_0q_2KrJ_m'(Kr)\}/(K^2r),\\ {}^1e_z^- &= AF_e(r),\\ {}^1h_r^- &= -A\{m\omega a_0F_e(r)+q_1KrJ_m'(Kr)\}/(K^2r),\\ {}^1h_\theta^+ &= A\{\omega a_0rF_e'(r)+(mq_1-\omega^2a_0b_1r^2)\,J_m(Kr)\}/(K^2r), \end{aligned}\right\} \tag{11·11}$$

and in these expressions

$$q_1 = m\left(\frac{\omega^2a_0b_0+{}^0p^2}{\omega^2a_0b_0-{}^0p^2}\right)\frac{b_1}{b_0}, \quad q_2 = \frac{2mb_1\,{}^0p}{b_0K^2}. \tag{11·12}$$

Equations (10·20) for ϕ, δ and $\bar\theta$ are now replaced by

$$\phi = {}^0pz-\omega t, \quad \delta = \tau b_1/b_0,$$

and

$$\bar\theta = \theta-\tau b_1z/b_0. \tag{11·13}$$

Using (6·7) and (6·8), (11·8) and (11·9) we obtain for the physical components of **E** and **H**

$$\left.\begin{aligned} E_r &= -l({}^0e_r^+-\tau\,{}^1e_r^+)\cos(m\bar\theta+\phi)+({}^0e_r^++\tau\,{}^1e_r^+)\cos(m\bar\theta-\phi),\\ E_\theta &= l({}^0e_\theta^--\tau\,{}^1e_\theta^-)\sin(m\bar\theta+\phi)-({}^0e_\theta^-+\tau\,{}^1e_\theta^-)\sin(m\bar\theta-\phi),\\ E_z &= \{l\sin(m\bar\theta+\phi)-\sin(m\bar\theta-\phi)\}\,\tau\,{}^1e_z^-, \end{aligned}\right\} \tag{11·14}$$

$$\left.\begin{aligned} H_r &= l({}^0h_r^--\tau\,{}^1h_r^-)\sin(m\bar\theta+\phi)-({}^0h_r^-+\tau\,{}^1h_r^-)\sin(m\bar\theta-\phi),\\ H_\theta &= -l({}^0h_\theta^+-\tau\,{}^1h_\theta^+)\cos(m\bar\theta+\phi)+({}^0h_\theta^++\tau\,{}^1h_\theta^+)\cos(m\bar\theta-\phi),\\ H_z &= \{l\cos(m\bar\theta+\phi)+\cos(m\bar\theta-\phi)\}\,{}^0h_z^+. \end{aligned}\right\} \tag{11·15}$$

If $l = 1$ these expressions reduce to

$$\left.\begin{aligned} E_r &= 2\{{}^0e_r^+\sin m\bar\theta\sin\phi+\tau\,{}^1e_r^+\cos m\bar\theta\cos\phi\},\\ E_\theta &= 2\{{}^0e_\theta^-\cos m\bar\theta\sin\phi-\tau\,{}^1e_\theta^-\sin m\bar\theta\cos\phi\},\\ E_z &= 2\tau\,{}^1e_z^-\cos m\bar\theta\sin\phi, \end{aligned}\right\} \tag{11·16}$$

$$\left.\begin{aligned} H_r &= 2\{{}^0h_r^-\cos m\bar\theta\sin\phi-\tau\,{}^1h_r^-\sin m\bar\theta\cos\phi\},\\ H_\theta &= 2\{{}^0h_\theta^+\sin m\bar\theta\sin\phi+\tau\,{}^1h_\theta^+\cos m\bar\theta\cos\phi\},\\ H_z &= 2\,{}^0h_z^+\cos m\bar\theta\cos\phi. \end{aligned}\right\} \tag{11·17}$$

Measurements of the rate of rotation of the fields for given amounts of torsion evidently yield, with the help of (6·5), (10·20) and (11·13) values for a_1 and b_1 if we can assume that the dielectric constant a_0 and the magnetic permeability b_0 for the undeformed material are known. Measurements of this kind for ferrite filled waveguides exhibiting the Faraday effect have been carried out by Hogan (1953).

SMALL DEPENDENCE UPON DEFORMATION

12. General formulation

We consider now the situation that arises when the angle of torsion τ is not necessarily small but the electromagnetic properties of the material do not depend greatly upon the deformation. In the limiting case, when this dependence is zero, the relations (2·1) reduce to

$$D^i = a_0 E^i, \quad B^i = b_0 H^i, \tag{12·1}$$

where a_0 and b_0 are constants. Otherwise, when this dependence is sufficiently small, the functions α_0, α_1 and α_2 occurring in (2·2) take the forms

$$\alpha_0 = a_0 + \nu\bar{\alpha}_0, \quad \alpha_1 = \nu\bar{\alpha}_1, \quad \alpha_2 = \nu\bar{\alpha}_2, \tag{12·2}$$

where a_0 is a constant, ν is a small real parameter which is independent of τ and r, and $\bar{\alpha}_0$, $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are polynomials in the invariants (3·8). The relations (3·9) may then be written as

$$\|\alpha^i_j\| = \begin{Vmatrix} a_0+\nu\bar{a}_1 & 0 & 0 \\ 0 & a_0+\nu\bar{a}_2 & \nu\bar{a}_4 \\ 0 & \nu r^2\bar{a}_4 & a_0+\nu\bar{a}_3 \end{Vmatrix}, \tag{12·3}$$

where

$$\left.\begin{aligned} \bar{a}_1 &= \bar{\alpha}_0 + \bar{\alpha}_1(r'^2-1) + \bar{\alpha}_2(r'^2-1)^2, \\ \bar{a}_2 &= \bar{\alpha}_0 + \tau^2 r^2[\bar{\alpha}_1 + \bar{\alpha}_2(1+\tau^2 r^2)], \\ \bar{a}_3 &= \bar{\alpha}_0 + \bar{\alpha}_2 \tau^2 r^2, \\ \bar{a}_4 &= \tau(\bar{\alpha}_1 + \bar{\alpha}_2 \tau^2 r^2) \end{aligned}\right\} \tag{12·4}$$

may be regarded as known functions of τ and r. Expressions corresponding to (12·3) apply for β^i_j with a_0 and $\bar{a}_\mu$ replaced by b_0 (constant) and $\bar{b}_\mu$ ($\mu = 1$ to 4). We note that we may, without loss of generality choose a_0, b_0 $\bar{a}_\mu$ and $\bar{b}_\mu$ in such a way that $\bar{a}_\mu$ and $\bar{b}_\mu$ vanish when the body is undeformed; a_0 and b_0 are then the dielectric constant and magnetic permeability respectively for the undeformed material.

Since p depends upon the deformation and on ν we write

$$p = {}^0p + \nu\,{}^1p + O(\nu^2), \tag{12·5}$$

where ${}^0p$ and ${}^1p$ are independent of ν and ${}^0p$ is the value of p for the undeformed material. Equations (4·9) and (4·11) then yield to the first order in ν

$$\Delta_1 = -K^2 + \nu\bar{\delta}_1, \quad \Delta_2 = -K^2 + \nu\bar{\delta}_2, \tag{12·6}$$

where

$$\left.\begin{aligned} \bar{\delta}_1 &= -\omega^2(a_0\bar{b}_2 + \bar{a}_1 b_0) + 2\,{}^0p\,{}^1p, \\ \bar{\delta}_2 &= -\omega^2(a_0\bar{b}_1 + \bar{a}_2 b_0) + 2\,{}^0p\,{}^1p, \end{aligned}\right\} \tag{12·7}$$

and K^2 is defined by (9·5).

We assume that to the first order in ν the field vectors $\mathbf{e}$ and $\mathbf{h}$ take the forms

$$\mathbf{e} = {}^0\mathbf{e} + \nu\,{}^1\mathbf{e}, \quad \mathbf{h} = {}^0\mathbf{h} + \nu\,{}^1\mathbf{h}, \tag{12·8}$$

with corresponding expansions for the components defined in (4·5).

The expansion of (4·8) and (4·10) in powers of ν yields the expressions (9·10) for ${}^0e_r$, ${}^0e_\theta$, ${}^0h_r$ and ${}^0h_\theta$, while from the coefficients of ν we obtain

$$\left.\begin{aligned}
{}^1e_r &= -\frac{1}{K^2r}\left\{\mathrm{i}\,{}^0pr\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r}-n\omega b_0\,{}^1h_z+\mathrm{i}\chi_1 r\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r}+\omega(\bar{b}_4\,{}^0pr^2-n\chi_6)\,{}^0h_z\right\},\\
{}^1e_\theta &= \frac{1}{K^2r}\left\{n\,{}^0p\,{}^1e_z+\mathrm{i}\omega b_0 r\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r}+(n\chi_2-\omega^2b_0\bar{a}_4r^2)\,{}^0e_z+\mathrm{i}\omega\chi_5 r\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r}\right\},\\
{}^1h_r &= -\frac{1}{K^2r}\left\{\mathrm{i}\,{}^0pr\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r}+n\omega a_0\,{}^1e_z-\omega(\bar{a}_4\,{}^0pr^2-n\chi_4)\,{}^0e_z+\mathrm{i}\chi_2 r\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r}\right\},\\
{}^1h_\theta &= \frac{1}{K^2r}\left\{n\,{}^0p\,{}^1h_z-\mathrm{i}\omega a_0 r\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r}+(n\chi_1-\omega^2a_0\bar{b}_4r^2)\,{}^0h_z-\mathrm{i}\omega\chi_3 r\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r}\right\},
\end{aligned}\right\}\tag{12·9}$$

where

$$\left.\begin{aligned}
\chi_1 &= {}^1p+\bar{\delta}_1\,{}^0p/K^2 = \{(\omega^2a_0b_0+{}^0p^2)\,{}^1p-\omega^2\,{}^0p(a_0\bar{b}_2+\bar{a}_1b_0)\}/K^2,\\
\chi_2 &= {}^1p+\bar{\delta}_2\,{}^0p/K^2 = \{(\omega^2a_0b_0+{}^0p^2)\,{}^1p-\omega^2\,{}^0p(a_0\bar{b}_1+\bar{a}_2b_0)\}/K^2,\\
\chi_3 &= \bar{a}_1+a_0\bar{\delta}_1/K^2 = -\{\omega^2a_0^2\bar{b}_2+{}^0p(\bar{a}_1\,{}^0p-2a_0\,{}^1p)\}/K^2,\\
\chi_4 &= \bar{a}_2+a_0\bar{\delta}_2/K^2 = -\{\omega^2a_0^2\bar{b}_1+{}^0p(\bar{a}_2\,{}^0p-2a_0\,{}^1p)\}/K^2,\\
\chi_5 &= \bar{b}_1+b_0\bar{\delta}_2/K^2 = -\{\omega^2b_0^2\bar{a}_2+{}^0p(\bar{b}_1\,{}^0p-2b_0\,{}^1p)\}/K^2,\\
\chi_6 &= \bar{b}_2+b_0\bar{\delta}_1/K^2 = -\{\omega^2b_0^2\bar{a}_1+{}^0p(\bar{b}_2\,{}^0p-2b_0\,{}^1p)\}/K^2.
\end{aligned}\right\}\tag{12·10}$$

When the coefficients (4·14) and (4·15) are expanded with the help of (12·3), (12·5) and (12·6) we obtain to the first order in ν

$$\left.\begin{aligned}
&A_1 = (\bar{A}_{11}+\nu\bar{A}_{12})\,r^2,\quad A_2=(\bar{A}_{11}+\nu\bar{A}_{22})\,r,\\
&A_3 = [\bar{A}_{11}(K^2-n^2/r^2)+\nu\bar{A}_{32}]\,r^2,\\
&D_1 = -(\bar{D}_{11}+\nu\bar{D}_{12})\,r^2,\quad D_2=-(\bar{D}_{11}+\nu\bar{D}_{22})\,r,\\
&D_3 = -[\bar{D}_{11}(K^2-n^2/r^2)+\nu\bar{D}_{32}]\,r^2,\\
&B_2 = -C_2=\nu\bar{B}_{21}r,\quad B_3=\nu\bar{B}_{31}r,\quad C_3=\nu\bar{C}_{31}r,
\end{aligned}\right\}\tag{12·11}$$

where

$$\left.\begin{aligned}
\bar{A}_{11} &= \omega a_0K^4,\quad \bar{A}_{12}=\omega K^2[\bar{a}_1K^2-a_0(\bar{\delta}_1+\bar{\delta}_2)],\\
\bar{A}_{22} &= \omega K^2\{\mathrm{d}[r(a_0\bar{\delta}_1+K^2\bar{a}_1)]/\mathrm{d}r-(2\bar{\delta}_1+\bar{\delta}_2)\,a_0\},\\
\bar{A}_{32} &= -\omega K^2\{2a_0\bar{\delta}_1(K^2-n^2/r^2)+K^2(n^2\bar{a}_2/r^2+a_0\bar{\delta}_2-K^2\bar{a}_3-2n\,{}^0p\bar{a}_4)\},\\
\bar{B}_{21} &= K^2\{\omega^2(a_0\bar{b}_4-b_0\bar{a}_4)\,K^2r^2-n\,{}^0p(\bar{\delta}_1-\bar{\delta}_2)\},\\
\bar{B}_{31} &= K^2\,\mathrm{d}(\omega^2a_0\bar{b}_4K^2r^2-n\,{}^0p\bar{\delta}_1)/\mathrm{d}r,
\end{aligned}\right\}\tag{12·12}$$

and $\bar{D}_{11}, \bar{D}_{12}, \bar{D}_{22}, \bar{D}_{32}$ and $\bar{C}_{31}$ are obtained from $\bar{A}_{11}, \bar{A}_{12}, \bar{A}_{22}, \bar{A}_{32}$ and $\bar{B}_{31}$ respectively by interchanging a and b and hence also $\bar{\delta}_1$ and $\bar{\delta}_2$.

When (12·11), together with the expansions for e_z and h_z are introduced into (4·12) and (4·13) we regain from the terms independent of ν the Bessel equations (9·8). The terms linear in ν yield similarly the equations

$$\left.\begin{aligned}
\frac{\mathrm{d}^2({}^1e_z)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\,{}^1e_z}{\mathrm{d}r}+\left(K^2-\frac{n^2}{r^2}\right){}^1e_z &= Q_1\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r}+(Q_2+2\,{}^0p\,{}^1p)\,{}^0e_z+\mathrm{i}\left(Q_3\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r}+Q_4\,{}^0h_z\right),\\
\frac{\mathrm{d}^2({}^1h_z)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\,{}^1h_z}{\mathrm{d}r}+\left(K^2-\frac{n^2}{r^2}\right){}^1h_z &= Q_5\frac{\mathrm{d}\,{}^0h_z}{\mathrm{d}r}+(Q_6+2\,{}^0p\,{}^1p)\,{}^0h_z+\mathrm{i}\left(Q_7\frac{\mathrm{d}\,{}^0e_z}{\mathrm{d}r}+Q_8\,{}^0e_z\right),
\end{aligned}\right\}\tag{12·13}$$

where

$$\left.\begin{aligned}
Q_1 &= \frac{d}{dr}(\omega^2 a_0^2 \bar{b}_2 + \bar{a}_1\, {}^0p^2)/(a_0 K^2) = -\frac{1}{a_0}\frac{d\chi_3}{dr},\\
Q_2 &= -\{\omega^2 a_0^2 \bar{b}_2 + \bar{a}_1\, {}^0p^2 + K^2 \bar{a}_3 + n^2[\omega^2 a_0^2(\bar{b}_1 - \bar{b}_2) - {}^0p^2(\bar{a}_1 - \bar{a}_2)]/(K^2 r^2) + 2n\, {}^0p \bar{a}_4\}/a_0,\\
Q_3 &= \omega\{(a_0 \bar{b}_4 - b_0 \bar{a}_4)\, r^2 - n\, {}^0p[a_0(\bar{b}_1 - \bar{b}_2) - b_0(\bar{a}_1 - \bar{a}_2)]/K^2\}/(a_0 r),\\
Q_4 &= \frac{\omega}{r}\frac{d}{dr}\left[\bar{b}_4 r^2 + \frac{n\, {}^0p}{a_0 K^2}(a_0 \bar{b}_2 + \bar{a}_1 b_0)\right],\\
Q_5 &= \frac{d}{dr}(\omega^2 b_0^2 \bar{a}_2 + \bar{b}_1\, {}^0p^2)/(b_0 K^2) = -\frac{1}{b_0}\frac{d\chi_5}{dr},\\
Q_6 &= -\{\omega^2 b_0^2 \bar{a}_2 + \bar{b}_1\, {}^0p^2 + K^2 \bar{b}_3 + n^2[\omega^2 b_0^2(\bar{a}_1 - \bar{a}_2) - {}^0p^2(\bar{b}_1 - \bar{b}_2)]/(K^2 r^2) + 2n\, {}^0p \bar{b}_4\}/b_0,\\
Q_7 &= a_0 Q_3/b_0,\\
Q_8 &= -\frac{\omega}{r}\frac{d}{dr}\left[\bar{a}_4 r^2 + \frac{n\, {}^0p}{b_0 K^2}(b_0 \bar{a}_2 + \bar{b}_1 a_0)\right].
\end{aligned}\right\} \quad (12\cdot14)$$

We observe that Q_μ are all functions of r which are independent of ${}^1p$.

13. Circular waveguide: transverse magnetic wave

We assume that propagation takes place along a solid circular rod of material in which the outer boundary $r = r_1$ is perfectly conducting. The analysis follows the lines of that of §§10 and 11 for the case of small torsion. The boundary conditions (10·1) and (10·2) continue to apply and we may consider modes of propagation based upon the transverse magnetic and transverse electric waves in the undeformed material.

For propagation based upon a transverse magnetic wave we obtain, as in §10 the solution

$$ {}^0e_z = {}^0e_z^+ = AJ_m(Kr), \quad {}^0h_z \equiv 0 \quad (m = |n|), \tag{13·1}$$

where A is a real constant and $$J_m(Kr_1) = 0. \tag{13·2}$$

From (13·1) and (12·13) it follows that

$$\left.\begin{aligned}
\frac{d^2({}^1e_z)}{dr^2} + \frac{1}{r}\frac{d\,{}^1e_z}{dr} + \left(K^2 - \frac{n^2}{r^2}\right){}^1e_z &= Af_e^+(r),\\
\frac{d^2({}^1h_z)}{dr^2} + \frac{1}{r}\frac{d\,{}^1h_z}{dr} + \left(K^2 - \frac{n^2}{r^2}\right){}^1h_z &= iAf_h^-(r),
\end{aligned}\right\} \quad (13\cdot3)$$

where

$$\left.\begin{aligned}
f_e^+(r) &= KQ_1 J_m'(Kr) + (Q_2 + 2\,{}^0p\,{}^1p)\, J_m(Kr),\\
f_h^-(r) &= KQ_7 J_m'(Kr) + Q_8 J_m(Kr).
\end{aligned}\right\} \quad (13\cdot4)$$

From (13·3) we obtain solutions analogous to that of §10 for the case of small torsion. Thus we have

$$\left.\begin{aligned}
{}^1e_z &= {}^1e_z^+ = AF_e^+(r),\\
{}^1h_z &= i\,{}^1h_z^-, \quad {}^1h_z^- = AF_h^-(r),
\end{aligned}\right\} \quad (13\cdot5)$$

where

$$\left.\begin{aligned}
F_e^+(r) = \tfrac{1}{2}\pi\Big\{Y_m(Kr)\int_0^r J_m(Ku) f_e^+(u)\,du &\\
- J_m(Kr)\int_0^r Y_m(Ku) f_e^+(u)\,u\,du + M_1 J_m(Kr)\Big\},&\\
F_h^-(r) = \tfrac{1}{2}\pi\Big\{Y_m(Kr)\int_0^r J_m(Ku) f_h^-(u)\,u\,du &\\
- J_m(Kr)\int_0^r Y_m(Ku) f_h^-(u)\,u\,du + M_2 J_m(Kr)\Big\},&
\end{aligned}\right\} \quad (13\cdot6)$$

M_1 and M_2 being arbitrary constants.

It can be shown that the expressions (13·6) and hence also the solutions (13·5), remain finite as $r \to 0$ provided the dependence of r on R is such that $r' = \mathrm{d}r/\mathrm{d}R$ can be expressed in the neighbourhood of $r = 0$ as a power series in r with a finite range of convergence.* This assumption implies that as $r \to 0$

$$r' = \kappa + O(r^{\gamma_1}), \tag{13·7}$$

where κ is a constant and γ_1 is a positive integer. For the invariants (3·8) we then have

$$\left.\begin{aligned} c^i_i &= (\kappa^2-1)+O(r^{\gamma_2}), \\ \tfrac{1}{2}(c^i_i c^j_j - c^i_j c^j_i) &= O(r^2), \quad |c^i_j| = O(r^{\gamma_3}) \quad (\gamma_3 \geqslant 2), \end{aligned}\right\} \tag{13·8}$$

where $\gamma_2 = 1$ if $\gamma_1 = 1$, and $\gamma_2 = 2$ if $\gamma_1 \geqslant 2$. Since $\bar{a}_0$, $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are polynomials in the invariants (3·8), it follows that as $r \to 0$

$$\bar{a}_\mu = \kappa_\mu + O(r^{\gamma_4}) \quad (\mu = 1 \text{ to } 4), \tag{13·9}$$

where κ_μ are constants and $\gamma_4 \geqslant \gamma_2$. A similar result applies for the functions $\bar{b}_\mu$. Since ω, a_0, b_0, n, ${}^0p$ and K are constants we infer from (12·14) and (13·9) that as $r \to 0$

$$\left.\begin{aligned} Q_1 &= O(r^{\gamma_4-1}), \quad Q_2 = O(r^{-2}), \\ Q_3 &= O(r^{-1}), \quad Q_4 = O(r^{\gamma_4-2}), \end{aligned}\right\} \tag{13·10}$$

and similarly

$$\left.\begin{aligned} Q_5 &= O(r^{\gamma_4-1}), \quad Q_6 = O(r^{-2}) \\ Q_7 &= O(r^{-1}), \quad Q_8 = O(r^{\gamma_4-2}). \end{aligned}\right\} \tag{13·11}$$

Hence, since $\gamma_4 \geqslant 1$, from (13·4)

$$f^+_e(r) = O(r^{\gamma_4+m-2}), \quad f^-_h(r) = O(r^{\gamma_4+m-2}) \tag{13·12}$$

as $r \to 0$. Since $m \geqslant 1$, the functions $F^+_e(r)$ and $F^-_h(r)$ defined by (13·6) remain finite as $r \to 0$.

In the expression (13·6) for $F^+_e(r)$, the term involving M_1 is of the same type as that appearing in the solution (13·1) for ${}^0e_z$ and may be incorporated into this part of the expansion for e_z. We may therefore replace the arbitrary constant A by $\bar{A} = A(1+\nu\pi M_1/2)$ in (13·1) and omit the last term from (13·6). The boundary conditions (10·2), which apply at $r = r_1$, are then satisfied if

$$Y_m(Kr_1)\int_0^{r_1} J_m(Ku) f^+_e(u)\,u\,\mathrm{d}u - J_m(Kr_1)\int_0^{r_1} Y_m(Ku) f^+_e(u)\,u\,\mathrm{d}u = 0, \tag{13·13}$$

and

$$Y'_m(Kr_1)\int_0^{r_1} J_m(Ku) f^-_h(u)\,u\,\mathrm{d}u - J'_m(Kr_1)\int_0^{r_1} Y_m(Ku) f^-_h(u)\,u\,\mathrm{d}u + M_2 J'_m(Kr_1) = 0. \tag{13·14}$$

Equation (13·14) determines M_2; (13·13) may be used to determine ${}^1p$ if we remember that the coefficients Q_μ defined by (12·14) are independent of ${}^1p$. It is convenient to write

$$Q_2 = \bar{Q}_2 - 2n\,{}^0p\bar{a}_4/a_0, \tag{13·15}$$

where, from (12·14)

$$\bar{Q}_2 = -\{(\omega^2 a_0^2 \bar{b}_2 + \bar{a}_1\,{}^0p^2 + K^2\bar{a}_3)\,K^2r^2 + n^2[\omega^2 a_0^2(\bar{b}_1-\bar{b}_2) - {}^0p^2(\bar{a}_1-\bar{a}_2)]\}/(a_0K^2r^2). \tag{13·16}$$

Remembering, from (9·5), that K^2 is independent of n, we see from (12·14) and (13·16) that Q_1 is also independent of n while $\bar{Q}_2$ is an even function of n. If we combine (13·15) with

* For an incompressible rod $r' = 1$.

(13·4) and introduce the resulting expression for $f_e^+(r)$ into (13·13), we obtain, on solving for ${}^1p$

$$ {}^1p = \mathscr{L}_e(r_1)/[2\,{}^0p\,\mathscr{F}_e(r_1)] - n\mathscr{M}_e(r_1)/[a_0\mathscr{F}_e(r_1)]. \tag{13·17}$$

In this expression

$$\mathscr{F}_e(r_1) = J_m(Kr_1)\int_0^{r_1} J_m(Ku)\,Y_m(Ku)\,u\,\mathrm{d}u - Y_m(Kr_1)\int_0^{r_1} J_m^2(Ku)\,u\,\mathrm{d}u, \tag{13·18}$$

and if we write

$$\Psi_e(r) = KQ_1 J_m'(Kr) + \overline{Q}_2 J_m(Kr), \tag{13·19}$$

then

$$\mathscr{L}_e(r_1) = Y_m(Kr_1)\int_0^{r_1} J_m(Ku)\,\Psi_e(u)\,u\,\mathrm{d}u - J_m(Kr_1)\int_0^{r_1} Y_m(Ku)\,\Psi_e(u)\,u\,\mathrm{d}u, \tag{13·20}$$

and

$$\mathscr{M}_e(r_1) = Y_m(Kr_1)\int_0^{r_1} \bar{a}_4 J_m^2(Ku)\,u\,\mathrm{d}u - J_m(Kr_1)\int_0^{r_1} \bar{a}_4 J_m(Ku)\,Y_m(Ku)\,u\,\mathrm{d}u. \tag{13·21}$$

The expression for $\mathscr{F}_e(r_1)$ may be simplified by using the result (see, for example, Watson, 1944, §5·11)

$$\int^r u\psi_n(Ku)\,\overline{\psi}_n(Ku)\,\mathrm{d}u$$
$$= \tfrac{1}{4}r^2\{2\psi_n(Kr)\,\overline{\psi}_n(Kr) - \psi_{n-1}(Kr)\,\overline{\psi}_{n+1}(Kr) - \psi_{n+1}(Kr)\,\overline{\psi}_{n-1}(Kr)\}, \tag{13·22}$$

where $\psi_n(Kr)$, $\overline{\psi}_n(Kr)$ each represent one or other of the Bessel functions $J_n(Kr)$, $Y_n(Kr)$. We obtain

$$\mathscr{F}_e(r_1) = \tfrac{1}{4}r_1^2\{2Y_m(Kr_1)\,J_{m-1}(Kr_1)J_{m+1}(Kr_1)$$
$$-J_m(Kr_1)\,[J_{m-1}(Kr_1)\,Y_{m+1}(Kr_1) + J_{m+1}(Kr_1)\,Y_{m-1}(Kr_1)]\}. \tag{13·23}$$

When K has been determined from (13·2), ${}^0p$ is obtained from (9·5) and since ${}^1p$ is given by (13·17), p follows immediately from (12·5). The quantities ${}^0e_z$, ${}^0h_z$, ${}^1e_z$ and ${}^1h_z$ are given by (13·1) and (13·5) and the remaining field variables are obtained by introducing these expressions into (9·10) and (12·9). Thus for the components of the vectors **e** and **h** we have

$$\left.\begin{aligned}(e_r, e_\theta, e_z) &= [\mathrm{i}({}^0e_r^- + \nu\,{}^1e_r^-),\quad {}^0e_\theta^+ + \nu\,{}^1e_\theta^+,\quad {}^0e_z^+ + \nu\,{}^1e_z^+],\\ (h_r, h_\theta, h_z) &= [{}^0h_r^+ + \nu\,{}^1h_r^+,\quad \mathrm{i}({}^0h_\theta^- + \nu\,{}^1h_\theta^-),\quad \mathrm{i}\nu\,{}^1h_z^-],\end{aligned}\right\} \tag{13·24}$$

where

$$\left.\begin{aligned}{}^0e_r^- &= -(A\,{}^0p/K)\,J_m'(Kr),\quad {}^0e_\theta^+ = [An\,{}^0p/(K^2r)]\,J_m(Kr),\quad {}^0e_z^+ = AJ_m(Kr),\\ {}^0h_r^+ &= -[An\omega a_0/(K^2r)]\,J_m(Kr),\quad {}^0h_\theta^- = -(A\omega a_0/K)\,J_m'(Kr),\end{aligned}\right\} \tag{13·25}$$

and

$$\left.\begin{aligned}{}^1e_r^- &= \frac{A}{K^2r}\left\{n\omega b_0 F_h^-(r) - {}^0pr\frac{\mathrm{d}F_e^+(r)}{\mathrm{d}r} - K\chi_1 rJ_m'(Kr)\right\},\\ {}^1e_\theta^+ &= \frac{A}{K^2r}\left\{n\,{}^0pF_e^+(r) - \omega b_0 r\frac{\mathrm{d}F_h^-(r)}{\mathrm{d}r} + (n\chi_2 - \omega^2 b_0\bar{a}_4 r^2)\,J_m(Kr)\right\},\\ {}^1e_z^+ &= AF_e^+(r),\\ {}^1h_r^+ &= \frac{A}{K^2r}\left\{{}^0pr\frac{\mathrm{d}F_h^-(r)}{\mathrm{d}r} - n\omega a_0 F_e^+(r) + \omega[{}^0p\bar{a}_4 r^2 - n\chi_4]\,J_m(Kr)\right\},\\ {}^1h_\theta^- &= \frac{A}{K^2r}\left\{n\,{}^0pF_h^-(r) - \omega a_0 r\frac{\mathrm{d}F_e^+(r)}{\mathrm{d}r} - \omega K\chi_3 rJ_m'(Kr)\right\},\\ {}^1h_z^- &= AF_h^-(r).\end{aligned}\right\} \tag{13·26}$$

We observe that when τ is sufficiently small, the coefficients $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, $\bar{b}_1$, $\bar{b}_2$ and $\bar{b}_3$, like the corresponding quantities in §9, are $O(\tau^2r^2)$ while to the first order in τr, $\bar{a}_4$ and $\bar{b}_4$ are constants. To the first order in τr, therefore,

$$\begin{gathered}
\chi_1 = \chi_2 = (\omega^2 a_0 b_0 + {}^0p^2)\, {}^1p/K^2, \\
\chi_3 = \chi_4 = 2a_0\, {}^0p\, {}^1p/K^2, \quad \chi_5 = \chi_6 = 2b_0\, {}^0p\, {}^1p/K^2, \\
Q_1 = Q_5 = 0, \quad Q_2 = -2n\, {}^0p\bar{a}_4/a_0, \quad Q_6 = -2n\, {}^0p\bar{b}_4/b_0, \\
a_0 Q_3 = b_0 Q_7 = \omega(a_0\bar{b}_4 - b_0\bar{a}_4)\, r, \\
Q_4 = 2\omega\bar{b}_4 \quad \text{and} \quad Q_8 = -2\omega\bar{a}_4
\end{gathered}$$

are all constants, and ${}^1p$ may be chosen in (13·4) to have the value $n\bar{a}_4/a_0$ so that $f_e^+(r)$, and hence also $F_e^+(r)$, vanishes. When then recover from (13·26) formulae equivalent to (10·14). To this order in τ, the quantities (13·26) are either odd or even functions of n.

When τ is not small, the behaviour of these functions under a reversal of the sign of n is less simple. In this case, the quantities defined in (13·18) to (13·21) are even functions of n and from (13·7) we obtain different values of ${}^1p$ by putting $n = m$ and $n = -m$ where m is a positive integer. Similarly, from (12·10), (12·14) and (13·4) to (13·6) we obtain different values for χ_μ, Q_2, Q_3, Q_4, Q_6, Q_7, Q_8, $F_e^+(r)$ and $F_e^-(r)$ for the cases $n = m$ and $n = -m$, but in general, none of these functions is either odd or even in n. The same is true of the components (13·26). The first-order quantities (13·25) are, of course, identical with those of §10.

As in the previous instance, the two waves defined for the values $n = m$, $n = -m$ may be combined to give a rotating wave. We write

$$\left.\begin{aligned}
&\mathbf{e} = \mathbf{e}_1 = (e_{1r}, e_{1\theta}, e_{1z}), \quad \mathbf{h} = \mathbf{h}_1 = (h_{1r}, h_{1\theta}, h_{1z}), \\
&{}^1p = {}^1p_1, \quad p = p_1 = {}^0p + \nu\, {}^1p_1 \quad \text{when} \quad n = m, \\
\text{and}\quad &\mathbf{e} = \mathbf{e}_2 = (e_{2r}, e_{2\theta}, e_{2z}), \quad \mathbf{h} = \mathbf{h}_2 = (h_{2r}, h_{2\theta}, h_{2z}), \\
&{}^1p = {}^1p_2, \quad p = p_2 = {}^0p + \nu\, {}^1p_2 \quad \text{when} \quad n = -m.
\end{aligned}\right\} \tag{13·27}$$

Similarly we denote the expressions (13·26) by

$$\begin{aligned}
&{}^1e_{1r}^-, \quad {}^1e_{1\theta}^+, \quad {}^1e_{1z}^+, \quad {}^1h_{1r}^+, \quad {}^1h_{1\theta}^-, \quad {}^1h_{1z}^- \quad \text{when} \quad n = m \\
\text{and by}\quad &-{}^1e_{2r}^-, \quad {}^1e_{2\theta}^+, \quad {}^1e_{2z}^+, \quad {}^1h_{2r}^+, \quad -{}^1h_{2\theta}^-, \quad {}^1h_{2z}^- \quad \text{when} \quad n = -m,
\end{aligned}$$

the signs affixed to the second set of quantities being chosen so that for sufficiently small τ, they differ only by terms of order τ^2r^2 from those of the first set. We allow ${}^0e_r^-$, ${}^0e_\theta^+$, ${}^0e_z^+$, ${}^0h_r^+$ and ${}^0h_\theta^-$ to represent the expressions obtained by writing $n = m$ in (13·25).

With these conventions (13·24) yields

$$\left.\begin{aligned}
(e_{1r}, e_{1\theta}, e_{1z}) &= [\mathrm{i}({}^0e_r^- + \nu\, {}^1e_{1r}^-), \quad {}^0e_\theta^+ + \nu\, {}^1e_{1\theta}^+, \quad {}^0e_z^+ + \nu\, {}^1e_{1z}^+], \\
(h_{1r}, h_{1\theta}, h_{1z}) &= [{}^0h_r^+ + \nu\, {}^1h_{1r}^+, \quad \mathrm{i}({}^0h_\theta^- + \nu\, {}^1h_{1\theta}^-), \quad \mathrm{i}\nu\, {}^1h_{1z}^-],
\end{aligned}\right\} \tag{13·28}$$

and when $n = -m$ we have

$$\left.\begin{aligned}
(e_{2r}, e_{2\theta}, e_{2z}) &= [\mathrm{i}l({}^0e_r^- - \nu\, {}^1e_{2r}^-), \quad -l({}^0e_\theta^+ - \nu\, {}^1e_{2\theta}^+), \quad l({}^0e_z^+ + \nu\, {}^1e_{2z}^+)], \\
(h_{2r}, h_{2\theta}, h_{2z}) &= [-l({}^0h_r^+ - \nu\, {}^1h_{2r}^+), \quad \mathrm{i}l({}^0h_\theta^- - \nu\, {}^1h_{2\theta}^-), \quad \mathrm{i}l\nu\, {}^1h_{2z}^-],
\end{aligned}\right\} \tag{13·29}$$

where l is an arbitrary constant, introduced as in §10 to allow for the two waves being of arbitrarily different amplitudes.

The physical components of **E** and **H** follow by combining these expressions with equations analogous to (6·7) and (6·8). We obtain

$$\left.\begin{aligned} E_r &= l({}^0e_r^- - \nu\,{}^1e_{2r}^-)\sin(m\bar\theta+\phi) - ({}^0e_r^- + \nu\,{}^1e_{1r}^-)\sin(m\bar\theta-\phi), \\ E_\theta &= -l({}^0e_\theta^+ - \nu\,{}^1e_{2\theta}^+)\cos(m\bar\theta+\phi) + ({}^0e_\theta^+ + \nu\,{}^1e_{1\theta}^+)\cos(m\bar\theta-\phi), \\ E_z &= l({}^0e_z^+ + \nu\,{}^1e_{2z}^+)\cos(m\bar\theta+\phi) + ({}^0e_z^+ + \nu\,{}^1e_{1z}^+)\cos(m\bar\theta-\phi), \end{aligned}\right\} \tag{13·30}$$

and

$$\left.\begin{aligned} H_r &= -l({}^0h_r^+ - \nu\,{}^1h_{2r}^+)\cos(m\bar\theta+\phi) + ({}^0h_r^+ + \nu\,{}^1h_{1r}^+)\cos(m\bar\theta-\phi), \\ H_\theta &= l({}^0h_\theta^- - \nu\,{}^1h_{2\theta}^-)\sin(m\bar\theta+\phi) - ({}^0h_\theta^- + \nu\,{}^1h_{1\theta}^-)\sin(m\bar\theta-\phi), \\ H_z &= \nu\{l\,{}^1h_{2z}^-\sin(m\bar\theta+\phi) - {}^1h_{1z}^-\sin(m\bar\theta-\phi)\}. \end{aligned}\right\} \tag{13·31}$$

From (6·5) and (13·27) the speed of propagation of the composite wave is

$$2\omega/(p_1+p_2) = 2\omega/[2\,{}^0p + \nu({}^1p_1 + {}^1p_2)], \tag{13·32}$$

and the rate of rotation per unit length of rod is

$$(p_1-p_2)/(2m) = \nu({}^1p_1 - {}^1p_2)/(2m). \tag{13·33}$$

In these expressions we have, from (13·17) and (13·27)

$$\left.\begin{aligned} {}^1p_1 + {}^1p_2 &= \mathscr{L}_e(r_1)/[{}^0p\,\mathscr{F}_e(r_1)], \\ {}^1p_1 - {}^1p_2 &= -2m\mathscr{M}_e(r_1)/[a_0\mathscr{F}_e(r_1)]. \end{aligned}\right\} \tag{13·34}$$

14. Circular waveguide: transverse electric wave

For propagation based on the transverse electric wave in the undeformed material, we choose, as in §11, the solution

$${}^0e_z \equiv 0, \quad {}^0h_z = {}^0h_z^+ = AJ_m(Kr), \quad (m = |n|) \tag{14·1}$$

of (9·8) with

$$J_m'(Kr_1) = 0. \tag{14·2}$$

Equations (12·13) then yield

$$\left.\begin{aligned} \frac{d^2({}^1e_z)}{dr^2} + \frac{1}{r}\frac{d({}^1e_z)}{dr} + \left(K^2 - \frac{n^2}{r^2}\right){}^1e_z &= iAf_e^-(r), \\ \frac{d^2({}^1h_z)}{dr^2} + \frac{1}{r}\frac{d({}^1h_z)}{dr} + \left(K^2 - \frac{n^2}{r^2}\right){}^1h_z &= Af_h^+(r), \end{aligned}\right\} \tag{14·3}$$

where

$$\left.\begin{aligned} f_e^-(r) &= KQ_3J_m'(Kr) + Q_4J_m(Kr), \\ f_h^+(r) &= KQ_5J_m'(Kr) + (Q_6 + 2\,{}^0p\,{}^1p)J_m(Kr). \end{aligned}\right\} \tag{14·4}$$

From (14·3) we obtain the solutions

$$\left.\begin{aligned} {}^1e_z = i\,{}^1e_z^-, \quad {}^1e_z^- &= AF_e^-(r), \\ {}^1h_z = {}^1h_z^+ &= AF_h^+(r), \end{aligned}\right\} \tag{14·5}$$

where

$$\left.\begin{aligned} F_e^-(r) &= \tfrac{1}{2}\pi\Big\{Y_m(Kr)\int_0^r J_m(Ku)f_e^-(u)\,u\,du \\ &\qquad - J_m(Kr)\int_0^r Y_m(Ku)f_e^-(u)\,u\,du + M_1'J_m(Kr)\Big\}, \\ F_h^+(r) &= \tfrac{1}{2}\pi\Big\{Y_m(Kr)\int_0^r J_m(Ku)f_h^+(u)\,u\,du \\ &\qquad - J_m(Kr)\int_0^r Y_m(Ku)f_h^+(u)\,u\,du + M_2'J_m(Kr)\Big\}. \end{aligned}\right\} \tag{14·6}$$

M_1' and M_2' being further arbitrary constants. The conditions (13·10) and (13·11) again ensure that these functions remain finite as $r \to 0$. The last term in the second of equations (14·6) may now be incorporated into the solution (14·1) for ${}^0h_z$, and we may therefore take M_2' as zero in the expression for $F_h^+(r)$. The boundary conditions (10·2) are then satisfied by taking

$$Y_m(Kr_1)\int_0^{r_1} J_m(Ku) f_e^-(u)\,u\,\mathrm{d}u - J_m(Kr_1)\int_0^{r} Y_m(Ku) f_e^-(u)\,u\,\mathrm{d}u + M_1' J_m(Kr_1) = 0, \quad (14·7)$$

$$Y_m'(Kr_1)\int_0^{r_1} J_m(Ku) f_h^+(u)\,u\,\mathrm{d}u - J_m'(Kr_1)\int_0^{r_1} Y_m(Ku) f_h^+(u)\,u\,\mathrm{d}u = 0. \quad (14·8)$$

Equation (14·7) determines M_1'; equation (14·8) may be used to determine ${}^1p$ and the procedure is exactly analogous to that of §13. We define $\overline{Q}_6$ by

$$\overline{Q}_6 = -\{(\omega^2 b_0^2 \bar{a}_2 + \bar{b}_1\,{}^0p^2 + K^2 \bar{b}_3)\, K^2 r^2 + n^2[\omega^2 b_0^2(\bar{a}_1 - \bar{a}_2) - {}^0p^2(\bar{b}_1 - \bar{b}_2)]\}/(b_0 K^2 r^2), \quad (14·9)$$

so that, from (12·14)

$$Q_6 = \overline{Q}_6 - 2n\,{}^0p\,\bar{b}_4/b_0. \quad (14·10)$$

If we write

$$\left.\begin{aligned}
\Psi_h(r) &= KQ_5 J_m'(Kr) + \overline{Q}_6 J_m(Kr),\\
\mathscr{L}_h(r_1) &= Y_m'(Kr_1)\int_0^{r_1} J_m(Ku)\,\Psi_h(u)\,u\,\mathrm{d}u - J_m'(Kr_1)\int_0^{r_1} Y_m(Ku)\,\Psi_h(u)\,u\,\mathrm{d}u,\\
\mathscr{M}_h(r_1) &= Y_m'(Kr_1)\int_0^{r_1} \bar{b}_4 J_m^2(Ku)\,u\,\mathrm{d}u - J_m'(Kr_1)\int_0^{r_1} \bar{b}_4 J_m(Ku)\,Y_m(Ku)\,u\,\mathrm{d}u,\\
\mathscr{F}_h(r_1) &= J_m'(Kr_1)\int_0^{r_1} J_m(Ku)\,Y_m(Ku)\,u\,\mathrm{d}u - Y_m'(Kr_1)\int_0^{r_1} J_m^2(Ku)\,u\,\mathrm{d}u,
\end{aligned}\right\} \quad (14·11)$$

then

$${}^1p = \mathscr{L}_h(r_1)/[2\,{}^0p\,\mathscr{F}_h(r_1)] - n\mathscr{M}_h(r_1)/[b_0\mathscr{F}_h(r_1)]. \quad (14·12)$$

In this result the expression for $\mathscr{F}_h(r_1)$ may be simplified by using (13·22) and the recurrence formula

$$2K\psi_m'(Kr) = \psi_{m-1}(Kr) - \psi_{m+1}(Kr) \quad (14·13)$$

for the Bessel functions $\psi_m(Kr)$. We obtain

$$\begin{aligned}
8K\mathscr{F}_h(r_1) = r_1^2\{2J_m Y_m(J_{m-1} - J_{m+1}) + (Y_{m+1} - Y_{m-1})\,(2J_m^2 - J_{m+1}J_{m-1})&\\
- J_{m-1}^2 Y_{m+1} + J_{m+1}^2 Y_{m-1}\},&
\end{aligned} \quad (14·14)$$

where

$$J_\mu = J_\mu(Kr_1) \quad \text{and} \quad Y_\mu = Y_\mu(Kr_1).$$

Equations (14·2) and (9·5) yield values for ${}^0p$, and p is obtained from (12·5). The quantities ${}^0e_z$, ${}^0h_z$, ${}^1e_z$ and ${}^1h_z$ are given by (14·1) and (14·5) and the remaining field variables again follow from (9·10) and (12·9). Thus, for the components of **e** and **h** we have

$$\left.\begin{aligned}
(e_r, e_\theta, e_z) &= [{}^0e_r^+ + \nu\,{}^1e_r^+,\quad \mathrm{i}({}^0e_\theta^- + \nu\,{}^1e_\theta^-),\quad \mathrm{i}\nu\,{}^1e_z^-],\\
(h_r, h_\theta, h_z) &= [\mathrm{i}({}^0h_r^- + \nu\,{}^1h_r^-),\quad {}^0h_\theta^+ + \nu\,{}^1h_\theta^+,\quad {}^0h_z^+ + \nu\,{}^1h_z^+],
\end{aligned}\right\} \quad (14·15)$$

where

$$\left.\begin{aligned}
{}^0e_r^+ &= [An\omega b_0/(K^2 r)]\,J_m(Kr), & {}^0e_\theta^- &= (A\omega b_0/K)\,J_m'(Kr),\\
{}^0h_r^- &= -(A\,{}^0p/K)\,J_m'(Kr), & {}^0h_\theta^+ &= [An\,{}^0p/(K^2 r)]\,J_m(Kr),\\
& & {}^0h_z^+ &= AJ_m(Kr),
\end{aligned}\right\} \quad (14·16)$$

$$\left.\begin{aligned}
{}^1e_r^+ &= \frac{A}{K^2 r}\left\{{}^0pr\frac{\mathrm{d}F_e^-(r)}{\mathrm{d}r}+n\omega b_0 F_h^+(r)-\omega({}^0p\bar{b}_4 r^2-n\chi_6)\,J_m(Kr)\right\},\\
{}^1e_\theta^- &= \frac{A}{K^2 r}\left\{n\,{}^0pF_e^-(r)+\omega b_0 r\frac{\mathrm{d}F_h^+(r)}{\mathrm{d}r}+\omega K\chi_5 rJ_m'(Kr)\right\},\\
{}^1e_z^- &= AF_e^-(r),\\
{}^1h_r^- &= -\frac{A}{K^2 r}\left\{{}^0pr\frac{\mathrm{d}F_h^+(r)}{\mathrm{d}r}+n\omega a_0 F_e^-(r)+K\chi_2 rJ_m'(Kr)\right\},\\
{}^1h_\theta^+ &= \frac{A}{K^2 r}\left\{n\,{}^0pF_h^+(r)+\omega a_0 r\frac{\mathrm{d}F_e^-(r)}{\mathrm{d}r}+(n\chi_1-\omega^2 a_0\bar{b}_4 r^2)\,J_m(Kr)\right\}\\
{}^1h_z^+ &= AF_h^+(r).
\end{aligned}\right\} \qquad (14\cdot17)$$

As before, the values $n=m$, $n=-m$ $(m=|n|)$ yield different values for ${}^1p$ and for the components of **e** and **h**, and combination of the two waves thus defined produces a rotating wave. For the components of these two waves we employ the notation of (13·27). We allow ${}^0e_r^+$, ${}^0e_\theta^-$, ${}^0h_r^-$, ${}^0h_\theta^+$ and ${}^0h_z^+$ to represent the expressions obtained by writing $n=m$ in (14·16) and denote the expressions (14·17) by

$${}^1e_{1r}^+,\quad {}^1e_{1\theta}^-,\quad {}^1e_{1z}^-,\quad {}^1h_{1r}^-,\quad {}^1h_{1\theta}^+,\quad {}^1h_{1z}^+\quad \text{when}\quad n=m$$

and
$${}^1e_{2r}^+,\quad -{}^1e_{2\theta}^-,\quad {}^1e_{2z}^-,\quad -{}^1h_{2r}^-,\quad h_{2\theta}^+,\quad h_{2z}^+\quad \text{when}\quad n=-m.$$

We then have

$$\left.\begin{aligned}
(e_{1r},e_{1\theta},e_{1z}) &= [{}^0e_r^+ +\nu\,{}^1e_{1r}^+,\quad \mathrm{i}({}^0e_\theta^- +\nu\,{}^1e_{1\theta}^-),\quad \mathrm{i}\nu\,{}^1e_{1z}^-],\\
(h_{1r},h_{1\theta},h_{1z}) &= [\mathrm{i}({}^0h_r^- +\nu\,{}^1h_{1r}^-),\quad {}^0h_\theta^+ +\nu\,{}^1h_{1\theta}^+,\quad {}^0h_z^+ +\nu\,{}^1h_{1z}^+],
\end{aligned}\right\} \qquad (14\cdot18)$$

and
$$\left.\begin{aligned}
(e_{2r},e_{2\theta},e_{2z}) &= [-l({}^0e_r^+ -\nu\,{}^1e_{2r}^+),\quad \mathrm{i}l({}^0e_\theta^- -\nu\,{}^1e_{2\theta}^-),\quad \mathrm{i}l\nu\,{}^1e_{2z}^-],\\
(h_{2r},h_{2\theta},h_{2z}) &= [\mathrm{i}l({}^0h_{2r}^- -\nu\,{}^1h_{2r}^-),\quad -l({}^0h_\theta^+ -\nu\,{}^1h_{2\theta}^+),\quad l({}^0h_z^+ +\nu\,{}^1h_{2z}^+)],
\end{aligned}\right\} \qquad (14\cdot19)$$

where l is again an arbitrary constant. From (6·7) and (6·8) the physical components of **E** and **H** become

$$\left.\begin{aligned}
E_r &= -l({}^0e_r^+ -\nu\,{}^1e_{2r}^+)\cos(m\bar\theta+\phi)+({}^0e_r^+ +\nu\,{}^1e_{1r}^+)\cos(m\bar\theta-\phi),\\
E_\theta &= l({}^0e_\theta^- -\nu\,{}^1e_{2\theta}^-)\sin(m\bar\theta+\phi)-({}^0e_\theta^- +\nu\,{}^1e_{1\theta}^-)\sin(m\bar\theta-\phi),\\
E_z &= \nu\{l\,{}^1e_{2z}^-\sin(m\bar\theta+\phi)-{}^1e_{1z}^-\sin(m\bar\theta-\phi)\},
\end{aligned}\right\} \qquad (14\cdot20)$$

and
$$\left.\begin{aligned}
H_r &= l({}^0h_r^- -\nu\,{}^1h_{2r}^-)\sin(m\bar\theta+\phi)-({}^0h_r^- +\nu\,{}^1h_{1r}^-)\sin(m\bar\theta-\phi),\\
H_\theta &= -l({}^0h_\theta^+ -\nu\,{}^1h_{2\theta}^+)\cos(m\bar\theta+\phi)+({}^0h_\theta^+ +\nu^1 h_{1\theta}^+)\cos(m\bar\theta-\phi),\\
H_z &= l({}^0h_z^+ +\nu\,{}^1h_{2z}^+)\cos(m\bar\theta+\phi)+({}^0h_z^+ +\nu\,{}^1h_{1z}^+)\cos(m\bar\theta-\phi).
\end{aligned}\right\} \qquad (14\cdot21)$$

Also, from (14·12) and (13·27)

$$\left.\begin{aligned}
{}^1p_1+{}^1p_2 &= \mathscr{L}_h(r_1)/[{}^0p\mathscr{F}_h(r_1)],\\
{}^1p_1-{}^1p_2 &= -2m\mathscr{M}_h(r_1)/[b_0\mathscr{F}_h(r_1)],
\end{aligned}\right\} \qquad (14\cdot22)$$

and the speed of propagation and rate of rotation of the resultant wave are obtained by inserting these expressions into (13·32) and (13·33).

The results presented in this paper were obtained in the course of research sponsored by the National Science Foundation.

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